COLLECTED GEOMETRICAL PAPERS

OF

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PREFACE

I have been induced to bring out a collected edition of my geometrical papers in order that they might be readily accessible to those who felt interested in them. Some of the publications in which they originally appeared, notably the Journal of the Asiatic Society of Bengal, are not readily accessible to European Mathematicians.

My New Methods in Geometry have evoked special interest in certain mathematical circles. I hope the publication of this collected edition of my papers will help to widen and multiply these circles. I have freely made curtailments of unessential portions in my original papers as well as small alterations and additions here and there, where by so doing there has been a gain in lucidity or rigour.

My best thanks are due to the authorities of the Calcutta University Press for kindly undertaking the publication.

CALCUTTA:
July, 1929.

S. MUKHOPADHYAYA.

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COLLECTED GEOMETRICAL PAPERS

GEOMETRICAL THEORY OF A PLANE NON-CYCLIC ARC FINITE AS WELL AS INFINITESIMAL[1]

BY

S. MUKHOPADHYAYA (1908)

INTRODUCTION.

The following paper is an attempt to study geometrically a plane convex arc, under the supposition that the radius of curvature exists at each point or that the radius of curvature as well as its first rate of variation exists. No complete geometry, however, has been attempted, the main object of the paper being to deduce a number of interesting theorems relating to an infinitesimal arc.

In the first place, consecutive points on a fixed curve have been defined as the intersections of the curve with a variable curve of given kind X, these consecutive points being only the position of ultimate coincidence of a number of real distinct points, which must have originally existed, in every case in the proximity of this position, separated by finite distances. The concept is a simple and natural one. In counting consecutive points the analyst, not infrequently, confounds real intersections with imaginary ones.

In the special case where a curve of given kind X, determinable uniquely by r distinct points, meets the curve in $r+1$ distinct points it is possible, under certain circumstances, to bring the $r+1$ points into coincidence, by varying the form and position of the curve of kind X. The method is a useful one and has been illustrated in Theorem I.

[1] From Journal, Asiatic Society of Bengal (New Series), Vol. IV, 1908.

Section 1.—Finite Arc.

A point O moving continuously with time, from a position P to position Q, describes a *line* PQ. If there is a tangent at each point of the line which turns continuously as O moves from P to Q along the line, then the line PQ will be called a *curve*. If the tangent turn continuously in the same direction the curve PQ will be called a *convex arc*, provided no straight line meets it at more than two points.

If a number of distinct points be determined on a convex arc PQ by intersection of a line of given kind X, and when their positions are varied by varying the line of given kind X, they approach a given point O and ultimately coincide with it, then in their final position they are called so many consecutive points at O, determined by the line of given kind X. Thus if X determine r consecutive points at O then in every double neighbourhood of O there must exist r distinct points on PQ through which a line of given kind X passes.

If a straight line pass through three consecutive points at O, then O is called a point of *inflexion*. Thus in every double neighbourhood of a point of inflexion there exist three distinct points lying on a straight line.

If a circle pass through four consecutive points at O, then O is called a *cyclic* point.

If the radius of the circle of curvature at a cyclic point be infinitely large then O is called a point of *undulation*. It is hardly justifiable to define a point of undulation as one where the tangent passes through four consecutive points. In the neighbourhood of a point of undulation four points on a straight line cannot exist whereas in such a neighbourhood four points on a circle always exist.

A convex arc will be called *cyclic* or *non-cyclic* according as there is or there is not a cyclic point in its interior.

In the convex arc discussed in this paper it will be supposed that the circle through any three points, distinct or consecutive, varies in a continuous manner as the points are shifted along the arc. The radius of the circle through any three distinct points will be always finite although in the limit when the three points coincide it may become zero or infinite. It will also be supposed that the rate of variation of the radius for the shifting of any one of the points is a continuous function of the positions of the points.

Theorem I.—No circle can meet a non-cyclic arc at more than three points.

If possible suppose a circle meets a non-cyclic convex arc at four distinct points, P, Q, R, S lying in order on the arc. Then by keeping P and S fixed and continuously varying the radius of the circle we can make Q and R come as close together as we choose. Again by keeping Q and R fixed and continuously varying the radius of the circle we can make P or S approach Q or R as close as we choose. By repeating the above two operations alternately a sufficient number of times, it is evident we can make P, Q, R, S come as close together as we like and ultimately coincide at some point O, lying between the initial positions of P and S. Thus there will be a cyclic point in the interior of the arc which is against hypothesis.

Cor.—If a circle meet a convex arc at four distinct points P, Q, R, S, then there must exist a cyclic point between P and S.

Theorem II.—If POQ be a non-cyclic arc, then angle POQ will continuously increase or decrease as O moves along the arc from P to Q.

If not, then two positions O_1 and O_2 can be found for O, between P and Q, such that angle PO_1Q is equal to angle PO_2Q. Therefore, P, O_1, O_2, Q are concyclic and there is a cyclic point between P and Q, which is against hypothesis.

Cor. A.—If the tangents PT and QT at P and Q are equal, then there must exist a cyclic point on the arc POQ. For, the angles TPQ and TQP, being the limiting values of the supplement of the angle POQ when O coincides with P and Q, respectively, cannot be equal in a non-cyclic arc.

Cor. B.—If the angle POQ continuously increase as O moves from P to Q, then the circle POQ will fall below the arc from P to O and above the arc from O to Q.

Def.—An arc POQ will be called *positive*, if the angle POQ continuously increase, as O moves from P to Q along the arc; and it will be called *negative*, if the angle POQ continuously decrease, as O moves from P to Q. If the arc POQ be positive then evidently the arc QOP is negative and *vice versa*.

Cor. C.—If the tangents at P and Q to a positive non-cyclic arc PQ, meet above the arc, then QT is greater than PT.

Theorem III.—If O be any point on a non-cyclic arc POQ, then the circle POO, passing through P and two consecutive points at O, will fall entirely below or above the given arc, according as the arc POQ is positive or negative.

In the first place, it is evident that the circle POO will lie entirely below or above the given arc, as it cannot intersect the arc at a fourth point.

Suppose the arc POQ is positive. Then the circle POO will fall entirely below the given arc.

Fig. 1

If not, let it lie entirely above, as represented by the dotted line (Fig. 1).

Take any point R on the given arc between P and O. Join QR and produce QR to meet the circle POO at S. Join PS, PR, PO and QO. Then evidently angle SPO is less than angle SQO, as Q falls inside the circle. Therefore angle PSQ is greater than angle POQ, which is contrary to hypothesis.

Similarly if the arc POQ be negative, then the circle POO will lie entirely outside the given arc.

The converse theorem is also evidently true, namely, the arc POQ will be positive or negative according as the circle POO falls continually entirely inside or outside the given arc, as O moves from P to Q.

Cor. A.—If POQ be a non-cyclic arc, then it will fall between the circles POO and QOO.

Cor. B.—If POQ be a positive non-cyclic arc, then the circle of curvature at P falls entirely within the circle of curvature at Q. Thus the radius of curvature at P is less than the radius of curvature at Q.

Theorem IV.—If POQ be a positive non-cyclic arc and S be any point in it, then the minor arcs PS and SQ will be also positive, *i.e.*, the angle POS will continuously increase as O moves from P to S, and the angle QOS will continuously increase as O moves from S to Q.

Join PS. Then since angle POQ continuously increases as O moves from P to Q, the circle POO continuously falls below the given arc. Hence as O moves from P to S, the circle POO falls below the arc PS, and hence the angle POS continuously increases as O moves from P to S.

Similarly, if O be taken in arc SQ it can be proved that the angle SOQ continuously decreases as O moves from Q to S, *i.e.*, the arc QS is negative. Therefore arc SQ is positive.

Cor. A.—If PQ be any positive non-cyclic arc, then any minor arc P′Q′ is also positive. For, PQ′ is positive, therefore P′Q′ is also positive.

Cor. B.—If in an arc POQ there be a cyclic point, then angle POQ cannot continuously increase or decrease as O moves from P to Q.

For, if there be a cyclic point S, on arc PQ, then in the neighbourhood of S, four distinct points, say, P′, R′, S′, Q′, must exist lying on a circle. Hence in the arc PQ, the angle POQ cannot continuously increase or decrease as O moves from P to Q. Hence in the arc POQ the angle POQ cannot continuously increase or decrease as O moves from P to Q, for then, by the method of the above theorem, the angle P′OQ′ would continuously increase or decrease as O moved from P′ to Q′.

Cor. C.—If in an arc POQ there be a cyclic point S, then a minor arc PSQ can always be found such that the tangents PT, QT at P, Q are equal.

For, in the neighbourhood of S, four distinct points P′, R′, S′, Q′ are obtainable lying on a circle. The point S will be between P′ and Q′. Keep R′S′ fixed and vary the circle till P′R′ or S′Q′ coincide. Then keep these latter coincident points fixed, and vary the circle till the other two points coincide.

Cor. D.—If POQ be a positive non-cyclic arc, then the radius of curvature at O continuously increases as O moves from P to Q.

Cor. E.—If in an arc POQ there be a cyclic point S, then the radius of curvature has a maximum or minimum value at S.

For, the circle of curvature at S as it passes through four consecutive points at S falls entirely above or below the arc at S. Thus if arc PS be positive, arc SQ will be negative and *vice versa*. The circles of curvature at P and Q will, therefore, both be less or both be greater than the circle of curvature at S.

Theorem V.—If POQ be a non-cyclic positive arc, and S any fixed point on it, then angle POS will continuously decrease as O moves from S to Q, and the angle QOS will continuously decrease as O moves from P to S.

If PRSQ be an infinitesimal arc, RS any minor chord parallel to PQ, and M, N the midpoints of PQ, RS, then the line through M, N, in its ultimate position is called the *deviation* [1] axis at P.

The angle between the normal and deviation axis at P, both drawn outwards, is called the angle of *aberrancy* at P.

Theorem VII.—In any convex infinitesimal arc POQ, the supplement θ of the angle POQ, and the angles α, β, which the tangents at P, Q make with PQ, are infinitesimals of the first order and ultimately equal.

For, if R, R_1, R_2 be the radii of the circles POQ, PPQ, PQQ, respectively, then R, R_1, R_2 are finite and ultimately equal to the radius of curvature at P.

But, $PQ=2R \sin \theta=2R_1 \sin \alpha=2R_2 \sin \beta$. Therefore, θ, α, β are ultimately equal infinitesimals of the first order.

Cor. A.—If PT and QT be tangents at P and Q then PT and QT are ultimately equal, and the radius r of the circle PQT is ultimately equal to half the radius of the circle of curvature at P.

Cor. B.—The difference between the arc PQ and chord PQ is less than a quantity which is an infinitesimal of the third order.

For, the convex arc PQ, falling inside the triangle PTQ, has length between PT+TQ and PQ. Hence the difference between the arc and chord is less than PT+TQ−PQ or $8r \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\alpha+\beta}{2}$ which is again less than $r\alpha\beta(\alpha+\beta)$.

Cor. C.—The difference between θ and $\sin \theta$ is less than a quantity which is an infinitesimal of the third order, θ being of the first order.

Theorem VIII.—The angle of aberrancy, at a cyclic point on a convex arc, vanishes.

Let O be a cyclic point. Take any infinitesimal arc POQ. Then, from Cor. C, Theorem IV, a smaller arc P′OQ′ can be always found, such that the tangents P′T and Q′T at P′ and Q′ are equal. Therefore, if R be the middle point of P′Q′, TR is at right angles to P′Q′.

[1] Transon introduced the term '*deviation axis*,' for which Salmon substituted '*aberrancy axis*.' Transon called tan δ the rate of deviation from circular form, an exceedingly suggestive expression, which Salmon cut down to 'aberrancy.' Both the terms have been retained, by the present writer, with a slight distinction in use. (See Liouville, Vol. VI, and Salmon's Higher Plane Curves, p. 368 3rd edition.)

Now, TR becomes the deviation axis at O, ultimately. Therefore, the deviation axis at O coincides with the normal at O, and the angle of aberrancy vanishes.

Theorem IX.—The partial rate of variation of the radius of curvature, at any point P of a non-cyclic arc, is $\tan\delta$, where δ is the angle of aberrancy at P.

Take an infinitesimal arc PRSQ, where RS is parallel to PQ. Then, from Theorem VI, we have $\tan\delta=\frac{UV}{PQ}$, where UV is the distance between the centres of the circles RSQ and PRS. Now, it is easily seen that UV is ultimately equal to the difference of the radii of the circles RSQ and PRS. Hence, $\tan\delta$ is equal to the partial rate of variation of the radius of curvature at P.

Cor. A.—If PQ be an infinitesimal non-cyclic arc, then the difference between the radii of the circles PQQ and PPQ is PQ $\tan\delta$, for, the circle PPQ is transformed into the circle PQQ by a single change of P into Q.

Cor. B.—The complete rate of variation of the radius of curvature at any point P, of a convex arc, is $3\tan\delta$, where δ is the angle of aberrancy at P (Transon's Theorem).

For, the complete variation of the circle of curvature PPP into QQQ, may be effected by three equal partial variations, *viz.*, that of P into Q three times repeated.[1]

Theorem X.—If PT, TQ be tangents at P and Q to a positive non-cyclic infinitesimal arc PQ, the difference of PT and TQ is ultimately equal to $2R\alpha^2\tan\delta$, where δ is the angle of aberrancy and R the radius of curvature at P, and α the angle TPQ.

For, if β be the angle PQT, then

$$\frac{PT}{TQ}=\frac{\sin\beta}{\sin\alpha}=\frac{\dfrac{PQ}{2\sin\alpha}}{\dfrac{PQ}{2\sin\beta}}=\frac{\text{radius of circle PPQ}}{\text{radius of circle PQQ}}$$

[1] The above simple and general demonstration of Transon's Theorem is based on the conception of partial rate of variation of curvature. Transon himself deduced his theorem from properties of conics (Liouville, Vol. VI).

Therefore,

$$\frac{TQ-PT}{TQ+PT}=\frac{\text{radius of PQQ}-\text{radius of circle PPQ}}{\text{radius of PQQ}+\text{radius of circle PPQ}}$$

or,
$$\frac{TQ-PT}{PQ}=\frac{PQ\tan\delta}{2R}, \text{ ultimately}$$

or,
$$TQ-PT=R\alpha^2\tan\delta.$$

Cor. A.—$\alpha-\beta=2\alpha^2\tan\delta$.

Theorem XI.—If O_1, O_2, O_3 be any three points on the positive non-cyclic infinitesimal arc POQ, then the radius of the circle $O_1O_2O_3$ is equal to $R\{1+2(\alpha_1+\alpha_2+\alpha_3)\tan\delta\}$, where α_1, α_2, α_3 are the angles which PO_1, PO_2, PO_3 make with the tangent at P, δ the angle of aberrancy and R the radius of curvature at P.

For, the radius of circle $O_1O_2O_3$ is evidently

$$R+(PO_1+PO_2+PO_3)\tan\delta=R+2R(\alpha_1+\alpha_2+\alpha_3)\tan\delta$$

since
$$2R=\frac{PO_1}{\alpha_1}=\frac{PO_2}{\alpha_2}=\frac{PO_3}{\alpha_3} \text{ in the limit.}$$

Theorem XII.—If s and l be the lengths of the arc and chord of any positive non-cyclic infinitesimal arc PQ, then $s=l=2R(\alpha+2\alpha^2\tan\delta)$, where δ is the angle of aberrancy and R the radius of curvature at P, and α the angle which the tangent at P makes with PQ.

For, if R' be the radius of the circle PPQ, then, by Theorem XI,

$$R'=R(1+2\alpha\tan\delta).$$

Therefore, chord $PQ=2R'\sin\alpha=2R'\alpha=2R(\alpha+2\alpha^2\tan\delta)$.

But the arc PQ differs from chord PQ by an infinitesimal of the third order.

Therefore, $s=l=2R(\alpha+2\alpha^2\tan\delta)$.

Theorem XIII.—If O_1, O_2, O_3 be any three points on the non-cyclic infinitesimal arc $PO_1O_2O_3Q$, the angle $O_1O_3O_2$ is equal to $(1-2\alpha_3\tan\delta)(\alpha_2-\alpha_1)$, where α_1, α_2, α_3 are the angles which PO_1, PO_2, PO_3 make with PT.

Let angle $O_1O_3O_2=x$.

Then, $\sin x = \dfrac{O_1O_2}{2R_{123}}$ and $\sin(\alpha_2-\alpha_1)=\dfrac{O_1O_2}{2R_{12}}$, where R_{123} and R_{12} mean the radii of the circles $O_1O_2O_3$ and PO_1O_2, respectively.

$$\text{Then, } \frac{\sin x}{\sin(\alpha_2-\alpha_1)}=\frac{R_{12}}{R_{123}}=\frac{R\{1+2(\alpha_1+\alpha_2)\tan\delta\}}{R\{1+2(\alpha_1+\alpha_2+\alpha_3)\tan\delta\}}$$

$$=1-2\alpha_3\tan\delta.$$

Therefore, $x=(\alpha_2-\alpha_1)\ (1-2\alpha_3\tan\delta)$.

Cor. A.—Angle $PO_2O_1=\alpha_1(1-2\alpha_2\tan\delta)$.

Theorem XIV.—In any non-cyclic infinitesimal arc PO_1O_2Q, chord $O_1O_2=(PO_2-PO_1)+R\alpha_1\alpha_2\,(\alpha_2-\alpha_1)$, neglecting infinitesimal of fifth order, where α_1, α_2 are the angles which PO_1, PO_2 make with the tangent at P, and R is the radius of curvature at P.

We have, by trigonometry,

$$O_1O_2+PO_1-PO_2=8R_{12}\sin\frac{O_1PO_2}{2}\sin\frac{O_1O_2P}{2}\sin\frac{O_1PO_2+O_1O_2P}{2}$$

But, $\qquad R_{12}=R(1+2(\alpha_1+\alpha_2)\tan\delta)$

$$\sin\frac{O_1PO_2}{2}=\sin\frac{\alpha_2-\alpha_1}{2}=\frac{\alpha_2-\alpha_1}{2}$$

$$\sin\frac{O_1O_2P}{2}=\frac{\alpha_1}{2}(1-2\alpha_2\tan\delta)$$

$$\sin\frac{O_1PO_2+O_1O_2P}{2}=\frac{\alpha_2-\alpha_1}{2}+\frac{\alpha_1}{2}(1-2\alpha_2\tan\delta)$$

$$=\frac{\alpha_2}{2}(1-2\alpha_1\tan\delta).$$

Therefore,

$$O_1O_2+PO_1-PO_2=R(\alpha_2-\alpha_1)\,\alpha_1\alpha_2\,(1+0\tan\delta)$$

$$=R(\alpha_2-\alpha_1)\,\alpha_1\alpha_2.$$

Theorem XV.—The difference $s-l$ between the lengths of arc and chord of an infinitesimal non-cyclic arc PQ is $\frac{1}{3}R\alpha^3$, neglecting infinitesimals of fifth order, where R is the radius of curvature at P, and α is the angle between chord PQ and the tangent at P.

Divide angle α into an infinite number of small parts (say n equal parts where n is large), by the lines PO_1, PO_2, PO_3, etc., where O_1, O_2, O_3, etc., are points on the arc PQ.

Then $s=\sum_1^n O_{r-1}\,O_r$ in the limit when $n=\infty$

$$l=\sum_r^n(PO_r-PO_{r-1})$$

Therefore, $s-l=\text{Lt}\sum_r^n(O_{r-1}\,O_r+PO_{r-1}-PO_r)$

$$=\text{Lt}\sum_1^n R(\alpha_r-\alpha_{r-1})\,\alpha_{r-1}\alpha_r$$

$$=\tfrac{1}{3}\,R\,\text{Lt}\sum_1^n\{\alpha^3_r-\alpha^3_{r-1}-(\alpha_r-\alpha_{r-1})^3\}$$

$$=\tfrac{1}{3}\,R\alpha^3-\tfrac{1}{3}\,R\,\text{Lt}\sum_1^n(\alpha_r-\alpha_{r-1})^3$$

$$=\tfrac{1}{3}\,R\alpha^3.$$

Since $\text{Lt}\sum_1^n(\alpha_r-\alpha_{r-1})^3=\text{Lt}\sum_1^n\left(\frac{r\alpha}{n}-\frac{(r-1)\alpha}{n}\right)^3=\text{Lt}\frac{\alpha^3}{n^2}=0.$

Cor. A.—The difference $s-l$ is independent of δ, if we neglect infinitesimals of fifth order, R and α being given.

Cor. B.—$\text{Sin}\,\theta=\theta-\dfrac{\theta^3}{6}$, neglecting infinitesimals of fifth order.

Cor. C.—Area of segment, bounded by s and l,

$=2R^2\Sigma\{(\alpha_r-\alpha_{r-1})\alpha_r\alpha_{r-1}+2\alpha_r\alpha_{r-1}(\alpha^3_r-\alpha^2_{r-1})\tan\delta\}$ (by Theorem XII)

$$=2R^2\{\tfrac{1}{3}\alpha^3+\alpha^4\tan\delta\}$$

For, $\qquad \Sigma(\alpha_r-\alpha_{r-1})\alpha_r\alpha_{r-1}=\tfrac{1}{3}\alpha^3$

and

$$\Sigma 2(\alpha^2_r-\alpha^2_{r-1})\alpha_r\alpha_{r-1}=\Sigma\{\alpha^4_r-\alpha^4_{r-1}-(\alpha_r-\alpha_{r-1})^3(\alpha_r+\alpha_{r-1})\}=\alpha^4$$

N.B.—If only the radius of curvature be finite and continuous and not also its partial rate of variation, then it is more easily shewn, by omitting $\tan\delta$, that $s-l$ is equal to $\tfrac{1}{3}\,R\alpha^3$, where we neglect infinitesimals of the *fourth* order, not *fifth*. The writer is not aware of these rigorous geometrical determinations having been made before. Text-book writers content themselves generally by stating that the difference is of the third order.

NEW METHODS IN THE GEOMETRY OF A PLANE ARC

I.—Cyclic and Sextactic Points [1]

BY

S. MUKHOPADHYAYA (1909)

The following paper introduces certain simple geometrical methods applicable to the general theory of plane curves. It brings into prominence a certain class of singular points, on a plane curve, to which, it would appear, sufficient attention has not been given hitherto. If we suppose n consecutive points to travel steadily along a given curve, and carry on their shoulders an osculating curve of a given kind, which varies continuously as it moves, then, upon the given curve, we shall usually have a number of places or singular points, where the osculating curve halts momentarily. At each halt, a change of shoulders is effected. The rearmost moves off and the foremost receives an accession, or, the foremost goes away and the rearmost is strengthened. For the moment the osculating curve is borne on $n+1$ shoulders, that is, by one shoulder more than would suffice to carry its full weight.

In the present paper two members of this class of singular points, the *cyclic* and the *sextactic*, have been studied together, more specially in relation to an *elementary convex oval*.

A *cyclic* point is a singular point, on a plane curve, where the circle of curvature passes through four consecutive points, instead of three. A *sextactic* point is a singular point, where the osculating conic passes through six consecutive points, instead of five. At a cyclic point, the circle of curvature may touch the given curve,

[1] From Bulletin of the Calcutta Mathematical Society, Vol. 1, 1909.

internally or externally. In the former case, the point will be called *in*-cyclic and in the latter case, *ex*-cyclic. Similarly, at a sextactic point, the osculating conic may touch the given curve internally or externally. In the former case it may be allowable to call the point *in*-sextactic, and in the latter case *ex*-sextactic. With this much of introduction, we may proceed to demonstrate a number of interesting propositions.

Prop. I.—If any circle meet a convex arc, at four points, O_1, O_2, O_3, O_4, then there must exist a *cyclic* point on the arc, between the two extreme points O_1 and O_4, but not coinciding with O_1 or O_4.

Prop. II.—If any conic meet a convex arc, at six points, O_1, O_2, O_3, O_4, O_5, O_6, then, there must exist a *sextactic* point on the arc, between the two extreme points O_1 and O_6, but not coinciding with O_1 or O_6.

In Proposition I, we shall suppose that the circle through any three points of the arc varies continuously, as the points are moved, in any manner, along the arc. This, of course, implies that the radius of curvature varies continuously but does not exclude the possibility of its becoming zero or infinite at a cyclic point.

In Proposition II, we shall suppose that the conic through any five points of the arc varies continuously as the points are moved, in any manner, along the arc. This conic must either be an ellipse, a parabola, or a hyperbola. In the last case, the five points of the arc must necessarily lie on the same branch of the hyperbola, for, five points, distributed on two different branches of a hyperbola cannot, evidently, lie on the same convex arc. For the purposes of this paper, we shall suppose that the conic through any five points of the arc, is always an ellipse, although this restriction is not *necessary* for Proposition II.

To prove Proposition I, it should be noticed that the four points O_1, O_2, O_3, O_4, determined by the intersection of a circle with the given arc, can be varied in position along the arc, by a continuous variation of the circle of intersection. Suppose we vary any two *adjacent* points O_1, O_2, by varying the circle, in such a way, that the remaining two points O_3, O_4, through which the circle passes, remain fixed. By this operation we can *draw together* O_1, O_2 as close as we like. When we thus draw together any two adjacent points O_1, O_2, it is to be understood, that they come indefinitely close, while O_3, O_4 remain fixed, but that they never overlap or cross each other or any

other point, *e.g.*, O_3 or O_4. The *order* of the points O_1, O_2, O_3, O_4, is therefore, strictly maintained. This will be obvious if we notice that circles through two fixed points cannot cross each other again.

Draw together, first, O_2, O_3 and then O_1, O_2 and then O_3, O_4, the remaining points, during each operation, continuing fixed. At the end of this cycle of three operations, O_1 and O_4 will have come closer together than at the beginning. By repeating this cycle a large number of times, we can bring the two extreme points O_1, O_4, as close together as we like, so that, ultimately, O_1, O_2, O_3, O_4 will have all come together at some point lying between the initial positions of O_1 and O_4. In fact, if O_1, O_2, O_3, O_4 do not come together ultimately, then there must be a minimum separation between O_1 and O_4. But this is impossible, for so long as the arc O_1O_4 is finite, it can be shortened by repeating the abovementioned cycle of operations by a finite quantity.

The ultimate point, where O_1, O_2, O_3, O_4 all come together, will be *in-cyclic* or *ex-cyclic*, according as the circle O_1,O_2,O_3,O_4 crosses in or crosses out at O_4, *initially*. This will be so, because the order of the points O_1, O_2, O_3, O_4, is maintained during each operation. It is possible, however, that during an operation, an extra pair of intersections, say, X, Y, may arise between a pair of adjacent points, say, between O_1 and O_2. In that case we may drop O_1 and X, and go on repeating our cycles on the shorter arc $YO_2O_3O_4$. Evidently the circle $O_1XYO_2O_3O_4$ will cross in or cross out at Y as it does at O_1. If an extra intersection, say Z, exist *beyond* the extremities O_1, O_4, then during the cycles of operation it will always move *further* beyond.

The proof of Proposition II is exactly similar, and similar observations apply to it. In this case, the cycle of operations may be described as follows: Draw together, first, O_3, O_4, and then, in succession, the pairs (O_2, O_3), (O_4, O_5), (O_1, O_2) and (O_5, O_6), the remaining four points, during each operation, continuing fixed. As conics through four points do not cross each other again, the order of the points O_1, O_2, O_3, O_4, O_5, O_6 will be strictly maintained during each operation.

Prop. III.—On any elementary oval, there must exist at least four cyclic points, two *in* and two *ex*.

Prop. IV.—On any elementary oval, there must exist at least six sextactic points, three *in* and three *ex*.

To prove Prop. III, draw a circle through any three points O_1, O_2, O_3 on the oval. This circle must intersect the oval again, in a fourth point O_4, as two closed figures intersect in an even number of points. Suppose the circle $O_1O_2O_3O_4$ crosses *in* and *out*, alternately, at O_1, O_2, O_3, O_4. To obtain the in-cyclic points, draw together O_2, O_3 at O_2-O_3, and O_4, O_1 at O_4-O_1, so that the circle O_2-O_3 O_4-O_1 has internal double contact with the oval at O_2-O_3 and O_4-O_1. There must now exist an in-cyclic point, in each of the arcs O_2-O_3 O_4-O_1 and O_4-O_1 O_2-O_3, by Proposition I. Thus two in-cyclic points are established. Similarly, if we draw together O_1, O_2 at O_1-O_2 and O_3, O_4 at O_3-O_4, we shall have an ex-cyclic point, in each of the arcs O_1-O_2 O_3-O_4 and O_3-O_4 O_1-O_2.

Prop. IV is proved in a similar way. Take any two equal parallel chords O_1O_2 and O_3O_4 in the oval. Then a conic through O_1, O_2, O_3, O_4 and any fifth point O_5 on the oval must be an ellipse, for, two equal parallel chords cannot lie in the same branch of a hyperbola. Let the ellipse through O_1, O_2, O_3, O_4, O_5 meet the oval again at a sixth point O_6, for, two closed figures must intersect at an even number of points. Suppose O_1, O_2, O_3, O_4, O_5, O_6 lie in order on the oval, and the conic through them crosses *in* and *out*, alternately, at O_1, O_2, O_3, O_4, O_5, O_6. To obtain the in-sextactic points, draw together O_1, O_2 at O_1-O_2, and then O_3, O_4, at O_3-O_4, and, finally, O_5, O_6, at O_5-O_6.

Then, the ellipse O_1-O_2 O_3-O_4 O_5-O_6 has internal triple contact, with the oval at O_1-O_2, O_3-O_4, O_5-O_6. Therefore, from Proposition II, we conclude, that there must be an in-sextactic point in each of the arcs O_1-O_2 O_3-O_4 O_5-O_6, O_3-O_4 O_5-O_6 O_1-O_2, O_5-O_6 O_1-O_2 O_3-O_4. Thus there will be, at least, two in-sextactic points on the oval. Let these two in-sextactic points be X, Y. Draw a narrow ellipse having internal double contact with the oval at X, Y. Let this ellipse grow, maintaining double contact with the oval at X, Y, till it touches the oval internally again at a third point Z, which may in special cases coincide with X or Y. Then, by Proposition II, there must be another in-sextactic point in the arc XZY. Thus three in-sextactic points are demonstrated. In exactly similar way, three ex-sextactic points on the oval can be proved.

The following six propositions refer to arcs which are either non-cyclic or non-sextactic. A non-cyclic arc is one which does not possess a cyclic point in it, except it may be at the extremities. A

non-sextactic arc is one which does not possess a sextactic point on it, except it may be at the extremities. On the non-cyclic arc we will suppose that the circle through any three points varies continuously. On the non-sextactic arc, we will suppose that the conic through any five points varies continuously and is, so far as this paper goes, always an ellipse.

Prop. V.—If O_1, O_2, O_3 be any three points in order, on a non-cyclic arc, then the radius of the circle $O_1O_2O_3$ will continuously increase (or decrease), if the points O_1, O_2, O_3 be shifted in any manner, along the arc, in the same direction, provided the order of the points be maintained and the angle $O_1O_2O_3$ be never less than a right angle.

Prop. VI.—If O_1, O_2, O_3, O_4, O_5 be any five points on a non-sextactic arc, then the area of the ellipse $O_1O_2O_3O_4O_5$ will continuously increase (or decrease), if the points be shifted, in any manner along the arc, in the same direction, provided the order of the points be maintained and the points be never so far separated from one another, that the elliptic arc $O_1O_2O_3O_4O_5$ exceeds the semi-ellipse.

To prove Proposition V, suppose the points O_1, O_2, O_3 are shifted, one by one, in order, along the arc, in the same direction. Then during the shifting of each point, the radius will continually increase (or decrease). If not, suppose, while O_2 is being shifted, O_1 and O_3 retaining their positions, the radius at first increases and then decreases, or at first decreases and then increases. Then O_2 will have two positions X, Y, between O_1, O_3, such that the radius of the circles O_1XO_3 and O_1YO_3 are equal. Therefore, we must have angles O_1XO_3, O_1YO_3 either equal or supplementary. But they cannot be supplementary, as then one of them will be acute, which is against hypothesis. Neither can the two angles be equal, for then the four points O_1, X, Y, O_3 would be con-cyclic and there would be a cyclic point on the given arc, which is also against hypothesis.

To prove Proposition VI, suppose the points O_1, O_2, O_3, O_4, O_5 are shifted in order, one by one, in the same direction, along the arc. Then during each shifting, the area of the ellipse O_1, O_2, O_3, O_4, O_5 will continually increase or decrease. If not, suppose, while any one point O_3 is being shifted, the others retaining their positions, the area at first increases and then decreases or at first decreases and then increases. Then O_3 will have two positions, X, Y, between

O_2 and O_4, for which the area is the same, that is, the area of the ellipse $O_1O_2XO_4O_5$ is equal to the area of the ellipse $O_1O_2YO_4O_5$. But it is easily shewn that the two areas cannot under the circumstances be equal (see following Lemma) unless the two ellipses coincide. Therefore O_1 O_2 X Y O_4 O_5 lie on the same ellipse, that is, there is a sextactic point on the given arc, which is against hypothesis.

Lemma.—If $O_1O_2XO_3O_4$ and $O_1O_2YO_3O_4$ be two eiliptic arcs, each less than the corresponding semi-ellipse in length, then the area of the first ellipse will be greater than that of the second ellipse, provided arc O_2XO_3 pass above the arc O_2YO_3.

Convert by orthogonal projection the first ellipse into a circle C and the second ellipse into another S. With same lettering, the arc O_2XO_3 of C will pass above the arc O_2YO_3 of S.

The semi-diameters of S which are parallel to O_1O_4 and O_2O_3, respectively, are equal as O_1O_4 and O_2O_3 are equally inclined to the axis of S. The semi-diameters conjugate to these are therefore also equal.

The centre of S falls below O_1O_4 as O_1YO_4 is less than a semi-ellipse. Hence the diameters of S which bisect O_2O_3 and are parallel to O_1O_4, respectively, both fall entirely within C. Thus each of two conjugate semi-diameters of S is each less than the radius of C, whence the theorem follows.

Prop. VII.—If O_1, O_2, O_3 be any three points, in order, on a non-cyclic arc, then the circle $O_1O_2O_3$ will always cross in at O_1 and O_3, or always cross out at O_1 and O_3, in whatever way we displace O_1, O_2, O_3, along the arc, maintaining their order.

Prop. VIII.—If O_1, O_2, O_3, O_4, O_5 be any five points on a non-sextactic arc, then the conic $O_1O_2O_3O_4O_5$ will always cross in at O_1 and O_5, or always cross out at O_1 and O_5, in whatever way we displace the points O_1, O_2, O_3, O_4, O_5, along the arc, maintaining their relative order.

The above two propositions hardly need a formal proof. In Proposition VII, the cutting in or cutting out at O_1 or O_3 can only be *screened*, if an extra point of intersection X arise beyond O_1 or O_3. But this is impossible as the arc is non-cyclic. Similar remarks apply to Proposition VIII.

Prop. IX.—If AB be a non-cyclic arc, in which any three points O_1, O_2, O_3 being taken, in order, the circle $O_1O_2O_3$ cuts in at O_1 and

O_3, then the circle of curvature at A falls entirely within the circle of curvature at B.[1]

Prop. X.—If AB be a non-sextactic arc, in which any five points O_1, O_2, O_3, O_4, O_5 being taken in order, the ellipse $O_1O_2O_3O_4O_5$ cuts in at O_1 and O_5, then the osculating ellipse at A falls entirely within the osculating ellipse at B.

To prove Proposition IX, move O_1, O_2, O_3, to A, so that we get the circle of curvature AAA at A, which falls below the arc AB. Similarly if we move O_3, O_2, O_1 to B we get the circle of curvature BBB at B which goes above the arc. Therefore, the circle AAA falls within the circle BBB, if we only consider portions above the chord AB. If we move O_3, O_2 to B and O_1 to A, we get the circle ABB, which falls below the arc and cuts AAA at some point C, above the chord AB. The circle ABB, which falls below the arc, touches at B the circle BBB, which goes above the arc. Therefore circle ABB falls within the circle BBB. Again the circle ABB cuts the circle AAA at A and C, therefore, below the chord AB, the circle AAA falls within the circle ABB, and, therefore, much more within the circle BBB. Thus the circle AAA falls within the circle BBB, both above and below the chord AB.

Analogous proof holds for Proposition X. Bring O_1, O_2, O_3, to A, and O_4, O_5 to B. Then ellipse AAABB falls below the given arc. If we bring down to A the other two points O_4, O_5, also, then the osculating ellipse AAAAA will fall below the given arc and cut the ellipse AAABB at some point C, above the chord AB but below the given arc. Therefore, below the chord AB, the osculating ellipse AAAAA falls within the ellipse AAABB, for these two ellipses have the four points A, A, A, C, common, and hence they cannot intersect again. Similarly, the ellipse AABBB goes above the arc and cuts the osculating ellipse BBBBB, which also goes above the arc, at some point D, above the arc. Therefore, below the chord AB, the ellipse AABBB falls within the ellipse BBBBB. But ellipses AAABB and AABBB have double contact at A and B, and the former goes below the arc and the latter above, therefore, the former

[1] It has been noticed before by P. G. Tait, and comes easily by *assuming* the shape of the evolute between two centres of curvature C and C′, for, if ρ and ρ' be the corresponding radii of curvature, then $\rho - \rho'$ is greater than the chord CC′ of the evolute (Scientific Papers of P. G. Tait, Vol. II, p. 403).

AAABB falls entirely within the latter AABBB. Hence below the chord AB, the ellipse AAAAA falls within the ellipse BBBBB. Also, since the former goes below the arc, and the latter above, therefore above the chord AB, the ellipse AAAAA falls within the ellipse BBBBB. Thus the osculating ellipse at A falls entirely within the osculating ellipse at B.

It may be pointed out that the director circle to the osculating ellipse at A falls entirely within the director circle to the osculating ellipse at B.

NEW METHODS IN THE GEOMETRY OF A PLANE ARC

II.—Cyclic Points and Normals [1]

BY

S. MUKHOPADHYAYA (1919)

INTRODUCTORY.

A *Convex Arc* for the purposes of this paper will be defined as follows :—

(*i*) It is a continuous curve bounded by two extreme points.

(*ii*) It has a tangent at each point and a positive sense along the tangent which turns continuously in the same direction.

(*iii*) No straight line meets it at more than two points.

(*iv*) The circle determined by any three points of the arc varies in a continuous manner with the determining points.

A *convex oval* may be defined as a closed curve of which every arc is convex.

The arc or oval will lie entirely on the right of each tangent taken in the positive sense. The positive sense along any three-pointic circle will be similarly defined.

An arc NPQ of a circle C intersecting a convex arc S at P will be said to *in-cross* S at P if it crosses from the convex to the concave side at P, and to *out-cross* S at P if it passes from the concave to the convex side.

A circle C is said to have *ordinary* contact with S at P if it passes through only two consecutive points of S at P. A circle having ordinary contact with S at P will be said to have *under*-contact with S at P if it falls on the concave side of S and to have *over*-contact with S at P if it falls on the convex side of S.

[1] From Bulletin of the Calcutta Mathematical Society, Vol. X, 1919.

A circle C passing through three consecutive points of S at P will be said to have *cross*-contact with S at P.

If NPQ be an arc of a circle having *cross*-contact with S at P then NPQ will be said to in-cross S at P or out-cross S at P according as NPQ passes from convex to concave or from concave to convex side of S at P. If NPQ in-crosses then we may say that the portion NP has over-contact and the portion PQ has under-contact with S at P.

If a circle C pass through four consecutive points of S at P then P is called a cyclic point of S and the circle C may be said to have cyclic contact with S at P. A cyclic point will be called *under-cyclic* or *over-cyclic* [1] according as the circle C falls on the concave or convex side of S at P.

We will denote an arc of S between P_1 and P_2 by S_{12} and an arc of C from P_1 to P_2 by C_{12}, and so on.

A circular arc C_{12} will be called *cyclic* to S_{12} if it meets S_{12} in two or more points besides P_1 and P_2. It will be either under- or over- or cross-cyclic to S. If it out-crosses S at P_1 and in-crosses S at P_2, it is under-cyclic to S_{12} and if it in-crosses S at P_1 and out-crosses S at P_2 it is over-cyclic to S_{12}. If C_{12} out-crosses S both at P_1 and P_2 or in-crosses S both at P_1 and P_2, it is cross-cyclic to S_{12}.

A fundamental theorem which has been established in my first paper, referred to, and of which we shall make frequent use in the present paper, may now be re-stated in the following form:

If a circular arc C_{12} is under-cyclic to S between P_1 and P_2, then there exists at least one under-cyclic point on S between P_1 and P_2. If a circular arc C_{12} is over-cyclic to S between P_1 and P_2, then there exists at least one over-cyclic point on S between P_1 and P_2. If a circular arc C_{12} is cross-cyclic to S between P_1 and P_2, then there exists at least one under-cyclic and one over-cyclic point on S between P_1 and P_2.

[1] In my first paper (Bulletin of the Calcutta Mathematical Society, Vol. 1, 1909) I have distinguished the two kinds of cyclic points and called them in-cyclic and ex-cyclic. The same two kinds have been called here under-cyclic and over-cyclic.

THEOREM I.

If P_1, P_2, P_3 be three points taken in order on a convex arc S and the normals at P_1, P_2, P_3 meet at a common point O, which is not the centre of curvature of S at P_2 and which is towards the concave side of S_{12}, then there exists at least one cyclic point X on S between P_1 and P_3 provided none of the angles P_1OP_2 and P_2OP_3 exceed two right angles. The point X will be under-cyclic or over-cyclic according as OP_2 is a maximal or minimal normal.

Case I.—When each of the angles P_1OP_2 and P_2OP_3 is less than two right angles.

We may suppose without any loss of generality that OP_1 and OP_3 are the two normals from O to S, nearest to OP_2 on either side, for, if X lie between the feet of two nearer normals on either sides, much more will it lie between the feet of two further normals on either sides.

Suppose OP_2 is a maximal normal. Then OP_2 is the maximum radius vector from O to S in the whole neighbourhood $P_1P_2P_3$, and is therefore greater than both OP_1 and OP_3. Draw a circle through P_1 to touch S at P_2. We will denote this circle by C and the arc of this circle from P_1 to P_2 by C_{12}. Then since P_1OP_2 is less than two right angles and OP_1 is less than OP_2 the arc C_{12} meets P_1O at an obtuse angle and therefore out-crosses S at P_1.

Similarly draw a circle C' through P_3 to touch S at P_2. Denote the arc of this circle from P_2 to P_3 by C'_{23}. Then C'_{23} will meet P_3O at an acute angle and therefore will in-cross S at P_3.

Then either C and C' will coincide or one will fall within the other.

If C and C' coincide then the circular arc $P_1P_2P_3$ will meet S under-cyclically between P_1 and P_3 and therefore there must exist at least one under-cyclic point on S between P_1 and P_3.

If C and C' do not coincide, then one will fall within the other.

The circle C will have either under-contact or over-contact or cross-contact with S at P_2.

If C has under-contact with S at P_2 then C_{12} must cross S_{12} somewhere between P_1 and P_2, and consequently C_{12} will meet S_{12} under-cyclically between P_1 and P_2. Thus there is an under-cyclic point on S between P_1 and P_2.

If C has over-contact with S at P_2 then C_{12} produced towards P_2 will pass between S_{23} and C'_{23}, *i.e.*, C will enter at P_2 the space bounded by S_{23} and C'_{23}. C must therefore come out of this space at some point P_4 on S_{23} between P_2 and P_3. Thus C meets S under-cyclically between P_1 and P_2.

If C has cross-contact with S at P_2 then C_{12} will either in-cross S at P_2 or out-cross S at P_2. In the former case there will be an under-cyclic point on S between P_1 and P_2, and in the latter case C will in-cross S_{23} at some point P_4 and there will be an under-cyclic point between P_2 and P_4.

Next suppose that OP_2 is a minimal normal. In this case we can prove, by reasoning exactly similar that there is at least one over-cyclic point on S between P_1 and P_3.

Case II.—When angle P_1OP_2 is less than two right angles and angle P_2OP_3 is equal to two right angles.

Suppose OP_2 is a minimal normal so that OP_1 and OP_3 are each greater than OP_2.

Draw a circle C to pass through P_1 and to touch S at P_2. Then because the angle P_1OP_2 is less than two right angles and OP_1 is greater than OP_2 the arc C_{12} will meet P_1O at an acute angle and consequently C_{12} will in-cross S at P_1.

The circle C will either have over-contact or under-contact or cross-contact with S at P_2. If C have over-contact with S at P_2 then C will cross S between P_1 and P_2 and consequently there will be an over-cyclic point on S between P_1 and P_2. If C have cross-contact with S at P_2 then C_{12} will either out-cross S at P_2 or in-cross S at P_2. In the latter case C_{12} must cross S between P_1 and P_2. So that in either case there will be an over-cyclic point on S between P_1 and P_2.

If C have under-contact with S at P_2 then C will either meet S_{23} between P_2 and P_3 at some point P_4 or fall below S_{23}. In the former case there is an over-cyclic point on S between P_1 and P_4.

In the latter case draw the circle C′ or rather the semi-circular arc C'_{23} to touch S at P_2 and P_3. If C'_{23} have over-contact with S at P_2 and P_3 then an over-cyclic point on S between P_2 and P_3 is assured. If C'_{23} have contacts over and under or under and over at P_2 and P_3 then C'_{23} must necessarily cross S between P_2 and P_3 and an over-cyclic point on S between P_2 and P_3 is assured.

If C'_{23} have under-contact with S at P_2 and P_3 then C' will enter the space formed by S_{12} and C_{12} at P_2 and consequently out-cross S at some point P_4 between P_1 and P_2. Consequently there will be an over-cyclic point on S between P_4 and P_3.

Thus on the supposition that OP_2 is a minimal normal there is always an over-cyclic point on S between P_1 and P_3.

If we had supposed OP_2 to be a maximal normal we could prove by similar reasoning that there is always an under-cyclic point on S between P_1 and P_3.

Corollary to Theorem I.

If the normals at P_1 and P_3 meet at P_2, then there is at least one over-cyclic point on S between P_1 and P_3. If the normals at P_1 and P_2 meet at P_3, then there is an over-cyclic or under-cyclic point on S between P_1 and P_3 according as P_2P_3 is a minimal or maximal normal.

Theorem II.

If OP_1 and OP_2 be two successive normals to a convex arc S from a point O, on the concave side of S, including between them an angle not exceeding two right angles, and if O be the centre of curvature of S at P_2, then there is at least one cyclic point on S between P_1 and P_2, which is under- or over- according as OP_1 is less or greater than OP_2.

Suppose angle P_1OP_2 is less than two right angles and OP_1 is less than OP_2.

Draw a circle C to pass through P_1 and touch S at P_2. Then the arc C_{12} of this circle will meet OP_1 at an acute angle and consequently out-cross S at P_1.

Draw a circle C' with centre O and radius OP_2. Then C' is the circle of curvature of S at P_2 and touches C externally at P_2 as OP_2 is greater than OP_1. The circular arc C_{12} will therefore have under-contact with S at P_2. Consequently C_{12} must in-cross S at some point P_3 between P_1 and P_2. Thus C_{12} is under-cyclic to S between P_1 and P_2 which ensures the existence of an under-cyclic point on S between P_1 and P_2.

If we suppose the angle P_1OP_2 to be equal to two right angles, then C_{12} will have under-contact with S at P_2 and either under- or over-contact with S at P_1. In the former case C_{12} is under-cyclic to S between P_1 and P_2 and in the latter case C_{12} is cross-

cyclic to S between P_1 and P_2. In either case the existence of an under-cyclic point on S between P_1 and P_2 is assured.

If OP_1 is greater than OP_2 the existence of an over-cyclic point on S between P_1 and P_2 can be similarly established.

In this theorem we have supposed O to be the centre of curvature of S at P_2. The centre of curvature of S at P_1 will in general be not at O but it can be at O as a special case.

Corollary to Theorem II.

If the centre of curvature of S at a point P_1 be a point P_2 which is on S then there is at least one under-cyclic point on S between P_1 and P_2.

The three following theorems follow at once from Theorems I and II and their corollaries:

Theorem III.

If from a point O on the concave side of a convex arc S it be possible to draw n normals to S, and if the angle between any pair of successive normals do not exceed two right angles, then there are at least $n-2$ cyclic points on S between the feet of the first and last normal.

Theorem IV.

If from a point O interior to a convex oval it be possible to draw n normals to the oval, and if the angle between any pair of successive normals do not exceed two right angles, then there are at least n cyclic points on the oval.

Theorem V.

If from a point O on a convex oval it be possible to draw n normals to S excluding the normal at O, then there are at least $n+1$ cyclic points on the oval.

If in the above theorems O be the centre of the circle of curvature at P for any normal OP, then such a normal has to be counted twice. If in addition the point P be a cyclic point, then the normal OP has to be counted thrice.

GENESIS OF AN ELEMENTARY ARC [1]

BY

S. MUKHOPADHYAYA (1926)

INTRODUCTORY.

The development of the *theory of elementary curves* is primarily due to *C. Juel* of Copenhagen. *P. Montel* has reviewed C. Juel's work in *the Bulletin des Sciences Mathematiques*, 1924, Part I, as also that of S. Mukhopadhyaya on similar lines. A bibliography on the subject occurs at the end of P. Montel's review.

C. Juel's concept of an elementary arc is exposed by P. Montel as follows:

"It is necessary above all, to define the simple element which serves as the basis for the construction of plane (elementary) curves, which we proceed in the first place to study with M. Juel. Let us imagine an arc of a continuous curve with extremities A and B; if this arc encloses with the chord AB, a convex domain, one can easily deduce from this the existence at each point of the arc of an anterior half-tangent and a posterior half-tangent. To this let us add the condition that these half-tangents have the same direction; our arc shall then possess, at each point, a tangent varying in a continuous manner with the point of contact. We shall thus obtain an *elementary arc*. Such an arc is met in two points at most by a straight line; one can draw to it two tangents at most from a point."

The definition of an elementary arc as outlined above, assumes that we know how to define a continuous curve in a satisfactory way—a thing which we perhaps do not know. The arc has undefined proportions and as such is of more limited use than the one defined in this paper.

The way in which an elementary arc has been evolved in this paper from a chain of cellular elements may prove interesting to geometers as a novel solution of the problem of the plane elementary arc on rigorous lines.

2. Consider an ordered set of a finite number of points A, P_1, P_2,....P_{n-1}, B in a restricted domain on a plane which may be Euclidean or non-Euclidean. The train of n sects AP_1, P_1P_2...... $P_{n-1}B$ constitutes a *linear chain of rank n*. The points A, P_1, P_2,

[1] From Bulletin of the Calcutta Mathematical Society, Vol. XVII, 1926.

......P_{n-1}, B will be supposed all distinct, except that B may coincide with A. In the latter case the chain is *closed* and in the former case the chain is *open*.

In the open linear chain of rank n there are n—1 vertices P_1, P_2,......P_{n-1} and two extremities A and B. In the closed linear chain of rank n there are n vertices and no extremities. The order A, P_1, P_2,......P_{n-1}, B will be called the *positive* order on the chain as distinguished from the order B, P_{n-1},......P_2, P_1, A which will be called the *negative* order on the chain.

Each of the sects AP_1, $P_1 P_2$,......P_{n-1} B will be called a *trace* of the chain. The trace PQ will be considered positive or negative according as P precedes or succeeds Q in the positive order on the chain. The extremities P and Q will be included in the trace PQ. Two consecutive traces PQ, QR can have only one point Q common unless they overlap. If no two non-consecutive traces have a common point and if two consecutive traces have only one point common, the chain will be called *simple*.

3. If PQ and QR be any two consecutive traces of a simple chain, P, Q, R being in positive order, then QR will be either to the right or to the left of PQ or in the prolongation of PQ. In the first case the chain will be said to have a *positive trend*, in the second a *negative trend* and in the third case a *zero trend*, at the vertex Q. The absolute amount of the trend at Q is measured by an angle less than two right angles between the directions of PQ and QR taken positively.

If a simple chain has at every vertex Q a trend of the same sign, with the possibility of a zero trend at some, the chain will be called *monocline* or of *unilateral trend*. A monocline chain may be either positively or negatively so, that is, it may be either of *dextro-lateral* or of *levo-lateral* trend.

A simple closed mono cline chain is called a *convex polygon*. We may suppose that in a convex polygon the trend does not vanish at any vertex, so that there are exactly n *bounding lines* in a convex polygon of rank n, consisting of the n traces of the simple closed mono-cline chain which defines it.

THEOREMS.

4. (i) *A convex polygon lies entirely on the same side of each of its bounding lines, that is if PQ be any bounding line, taken in the positive sense, all the other bounding lines will fall on the right side or left side of PQ according as the polygon is positively or negatively monocline, respectively.*

(ii) *No straight line which does not pass through two consecutive vertices can meet a convex polygon at more than two distinct points.*

It is usual to assume *Theorem* (i) as the distinguishing property of a convex polygon and to deduce *Theorem* (ii) from it. Theorem (i) however can be proved from definition of a convex polygon as follows:

Suppose, if possible, that such a polygon lies partly on one side and partly on the other side of a bounding line PQ. Suppose NP and QR are respectively the bounding lines which immediately precede and succeed PQ, N, P, Q, R being in positive order on the polygon, which we will suppose, has a dextro-lateral trend. Then NP and QR lie on the right side of PQ, but as part of the polygon lies to the left of PQ by hypothesis, PQ meets the polygon again at some point X. Suppose X lies on PQ produced towards Q, so that the part of the polygon between Q and X lies wholly to the right of PQ, as QR is to the right of PQ. Turn QX about Q towards the right till QX falls along QR. Then X will either coincide with R or have a distinct position X_1 on QR produced towards R. In the former case, suppose RS is the bounding line immediately succeeding QR, so that X finally travels along RS to reach R. RS is therefore to the left of QR whereas PQ is to the right of QR, which is impossible as the polygon has a unilateral trend.

In the latter case, turn RX_1 again to the right till RX_1 falls along RS. Then X_1 will either coincide with S or will have a distinct position on RS produced towards S.

The former is impossible and the latter leads to the repetition of the process of rotation to the right. But the number of vertices of the polygon which may lie between R and X is finite and consequently the number of possible rotations to the right will soon be exhausted, rendering the alternative position of X impossible. Thus *Theorem* (i) cannot be false.

To prove *Theorem* (ii), suppose, if possible, a straight line other than a bounding line meets the polygon at three distinct points U, V, W, in positive order on the polygon. Then V must also lie between U and W on the straight line UW as the polygonal chain is simple. Suppose V is an interior point or end-point of the bounding line PQ, so that U and W lie on opposite sides of PQ. The portions UP and QW of the polygon will therefore lie wholly or partly on opposite sides of PQ. This contradicts *Theorem* (i).

5. Consider a simple chain $A\,P_1\,P_2\ldots\ldots P_{n-1}\,B$ of unilateral trend all of whose vertices lie on the same side of AB. Such a chain may be called a *convex chain*.

Suppose all the vertices of a convex chain $AP_1P_2\ldots\ldots P_{n-1}B$ are *interior* points of a triangle ATB such that the angle between AT produced and TB is less than a given acute angle a. Also suppose AB is less than a certain length l, so that the exterior angle theorem holds for the domain enclosed by the triangle. The tringle ATB will be called the *principal cell* of the chain and the chain $AP_1 P_2\ldots\ldots P_{n-1}B$ will be called *an elementary chain* of cell-angle$<a$ and base $AB<l$.

If NP, PQ, QR be any three consecutive traces of an elementary chain in cell ATB then each of these traces produced positively will meet TB and produced negatively will meet AT. Consequently NP produced positively and QR produced negatively will intersect at some point U interior to the triangle ATB, such that the angle between the positive directions of NP and QR is less than a. Also $PQ<AB<l$, for if PQ produced meets AT at U and BT at W, then

$$PQ<VW<VB<AB.$$

The triangle PUQ will be called an *elementary cell* on trace PQ or carried by trace PQ of the elementary chain $AP_1P_2\ldots\ldots\ldots P_{n-1}B$.

The elementary cell on initial trace AP_1 will be a triangle AXP_1 where X is the intersection of P_1P_2 produced negatively and a line AX which lies between AT and AP_1 and determined in any consistent manner. Similarly the elementary cell on final trace $P_{n-1}B$ is a triangle $\bar{B}YP_{n-1}$ where Y is the intersection of $P_{n-2}P_{n-1}$ produced positively and a line BY which lies between BP_{n-1} and BT and determined in any consistent manner.

The elementary cells carried by the successive traces of a given elementary linear chain form an *elementary cellular chain* carried by a given elementary linear chain. It may be observed that each elementary cell falls entirely inside the principal cell of the chain with the exception of the initial and final elementary cells which have a corner at A and B respectively. If PQ and RS are two non-adjacent traces of the elementary chain then the corresponding elementary cells will lie entirely outside one another.

6. The length of the longest trace of a linear chain will be called the *head* of the traces and that of the shortest trace will be called the *tail* of the traces. The magnitude of the largest of the elementary

cell-angles of a cellular chain will be called the *head* of the cell angles and that of the smallest of the cell-angles will be called the *tail* of the cell-angles.

If the rank of a given elementary chain c be increased by the interpolations of additional vertices between pairs of consecutive vertices of the given chain and the new chain c' thus obtained be also elementary, then c' will be called a *gemmatic extension* of c or gemmatically derived from c, *provided*

(*i*) the order of the vertices of c is the same in c and c'

(*ii*) the extremities of c and c' are the same ;

(*iii*) the principal cell AT'B of c' is the same as the principal cell of c or falls within it ;

(*iv*) the initial and final elementary cells of c' fall within the initial and final elementary cell respectively of c with the points A and B respectively common.

If PQ be a trace of c and P'Q' of c' such that the vertices P', Q' of c' fall between P, Q, *i.e.*, the points P, P', Q', Q are vertices of c' in order, then P'Q' is said to have been gemmatically derived from PQ. P' may however coincide with P or Q' with Q. The elementary cell carried by P'Q' in c' is also said to have been gemmatically derived from the elementary cell carried by PQ in c.

7. A system of elementary linear chains $c_1, c_2, \ldots\ldots\ldots, c_r, \ldots\ldots\ldots$ such that each chain except the first is gemmatically derived from the one just preceding it, will be called a *gemmatic system of elementary chains.*

Similarly a system of cellular chains carried by a gemmatic system of elementary linear chains will be also called gemmatic.

Each of the above two systems will be called *regular* if the heads of the traces of $c_1, c_2 \ldots\ldots\ldots c_r, \ldots\ldots\ldots$ form a monotone decreasing sequence of zero limit and the heads of the elementary cell-angles of $c_1, c_2, \ldots c_r, \ldots$ also form a monotone sequence of zero limit.

A sequence of traces $t_1, t_2, \ldots\ldots\ldots t_r, \ldots\ldots\ldots$ belonging respectively to chains $c_1, c_2, \ldots\ldots\ldots, c_r, \ldots\ldots\ldots$ of a regular gemmatic system which are such that each except the first is gemmatically derived from the one just preceding it, will be called a *regular gemmatic sequence of traces.* The corresponding elementary cells belonging to $c_1, c_2, \ldots\ldots\ldots, c_r, \ldots\ldots\ldots$ respectively, will be called a *regular gemmatic sequence of cells.* A regular gemmatic sequence of cells will necessarily have a unique limiting point which is also

the limiting point of the corresponding regular gemmatic sequence of traces.

If PQ and RS be two non-adjacent traces of an elementary chain c, the corresponding elementary cells of c will entirely lie outside each other with no point common and consequently the limiting points of any two regular gemmatic sequences of cells derived from them will be entirely distinct.

An *elementary arc* may now be defined as the aggregate of limiting points of all possible regular gemmatic sequences of elementary cells $e_1, e_2, \ldots, e_r, \ldots$ belonging respectively to a regular gemmatic system of cellular chains, $c_1, c_2, \ldots, c_r, \ldots$ More briefly an elementary arc may be defined as the limit of a regular gemmatic system of cellular chains.

8. The following properties of an elementary linear chain are evident:

(i) No straight line other than one passing through two consecutive vertices can meet an elementary chain closed by its base AB at more than two points.

(ii) The successive traces $AP_1, P_1P_2, \ldots P_{n-1}B$ of an elementary chain meet when produced negatively and positively the sides AT and TB respectively of its principal cell at two ordered rows of points $A, U_1, U_2, \ldots, U_{n-1}$ and $V_{n-1}, V_{n-2}, \ldots V_1, B$.

(iii) Every part of an elementary chain is an elementary chain.

(iv) If a point P travels continuously from A to B along the chain, the distance AP continuously increases and distance BP continuously diminishes.

The corresponding properties of an elementary arc may be rigorously deduced:

(i) No straight line can meet an elementary arc in more than two points.

(ii) There exists a tangent at each point P of an elementary arc which changes its direction continuously in the same sense as P travels from A to B along the arc.

(iii) Every part of an elementary arc is an elementary arc.

(iv) If a point P travels continuously from A to B along the arc, the distance AP continuously increases and the distance BP continuously diminishes.

GENERALIZED FORM OF BOHMER'S THEOREM FOR AN ELLIPTICALLY CURLED NON-ANALYTIC OVAL[1]

BY

S. Mukhopadhyaya

1

Def. (*i*).—A *convex oval* V has the fundamental property that any n distinct points on it determine a unique convex n-gon of which they are the vertices and the order of the vertices of the n-gon is the order of the points on V.

There is a *positive* order on V and a *negative* order which is its reverse. If P_1, P_2, P_3 be in positive order on V, then P_3 lies on the right of the line P_1P_2. If P_1, P_2, P_3 be in positve order, we shall simply say they are *in order*.

The convex *non-analytic* oval discussed in this paper is of the class *elementary*. It consists of a closed continuous curve having a positive sense along it determined by the positive order of the points upon it. There is a *unique tangent* at each point P and a positive sense along the tangent such that every other point of the oval lies always on the right of the tangent. The tangent turns *continuously* on the right as one proceeds in the positive sense along the oval. Such an oval is obviously *rectifiable*.

Any point of the plane which lies to the right of every tangent and is not a point of the oval itself is an *interior* point of the oval. Any straight line through an interior point of the oval meets the oval at two points.[2]

[1] P. Böhmer in an elegant paper, published in the *Mathematischen Annalen*, Vol. 60, pp. 256-63, 1905, was the first to prove that for an *analytic oval*, in which the osculating conic at each point is an ellipse, the conic through any five points is also an ellipse. The methods employed by him are by themselves quite interesting, specially the use he makes of the *curvature form*.

[2] *Vide Genesis of an Elementary Arc*, by S. Mukhopadhyaya, Bull. Cal. Math. Soc., Vol. 18, 1926, pp. 153-68.

The only other lines, besides straight, which will be discussed in this paper, in connection with their intersections with the oval are *conics*.

2

The oval will not be postulated to possess a *definite* circle of curvature at any given point.

If a given conic S meets the oval V at P, it will be supposed to meet V at a *definite one-point* at P, provided it crosses V at P but does not touch, and at a definite two-point at P, if it touches V at P but does not cross. In the latter case there is either an internal contact (under-contact) or an external contact (over-contact) of S with V at P.

If S touches V at P as well as crosses, then we will say that three points at V are *associated* by S at P. Two of these are definite points of V and the third, we will say, is a *possible* point of V. The two *definite* points of V at P associated with the third *possible* point of P at V will determine a *possible circle of curvature* of V at P. This possible circle of curvature of V at P agrees with the circle of curvature of the given conic S at P. Two conics S and S′, each of which has cross-contact with V at P may have different curvatures at P.

Def. (*ii*).—By a *definite five-pointic conic* of V will be understood a conic which passes through five definite points of V.

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A point P on V may be determined by its arcual distances from a fixed point on V, measured positively in the positive sense along V.

Def. (*iii*).—A point P (s) on V will be called *elliptic* if a finite neighbourhood $(s-d, s+d)$ of P (s) exists such that every definite five-pointic conic S of this neighbourhood is an ellipse.

Def. (*iv*).—An elementary convex oval V will be called *elliptically curled* if every point P of V is elliptic in the sense above defined.

Böhmer's Theorem can now be stated for the above oval as:

(A) Every definite five-pointic conic of an elliptically curled oval is an ellipse.

A more general form of the above theorem is:

(B) If every *hexadic point* of an elementary convex oval be elliptic then every definite five-pointic conic of V is an ellipse.

The definition of a hexadic point will be given later. See under Cor. ii, Lemma VI.

It will appear from our investigations that every convex oval possesses some hexadic points. If every point on V is elliptic, then the hexadic points must necessarily be elliptic and Böhmer's Theorem (A) follows at once from the more general form (B).

We will proceed to establish Theorem (B). For this purpose it will be necessary to establish a number of useful *Lemmas*.

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Def. (v).—A range R_n of n $(n>3)$ distinct points $P_1, P_2, P_3 \ldots\ldots P_n$ on an ellipse, parabola or single branch of a hyperbola, will be said to be in order, if they are the successive vertices of a convex n-gon. The order will be positive if P_3 lie on the right of P_1P_2. A range in positive order on a conic will be simply called a range in order on the conic.

If P_1, P_2, P_3 be three points on a branch S of a hyperbola and Ω, Ω' be the two points at infinity on S and if $\Omega, P_1, P_2, P_3, \Omega'$ be in order, then $P_3, \Omega', \Omega, P_1$ are also in order, so that each of the points Ω and Ω' lies between P_3 and P_1, whereas P_2 lies between P_1 and P_3.

If $\Omega, P_1, P_2, P_3, \Omega'$ be in order on a branch S of a hyperbola and P_4 lie on the other branch S_0 then we will say that P_4 lies between P_3 and P_1 on (S, S_0). The branch S_0 will be called the *obverse* of S. In this case $P_3, \Omega', P_4, \Omega, P_1$ are in order on (S, S_0), as also P_1, P_2, P_3, P_4.

It should be noted that although we say that the points P_1, P_2, P_3, P_4 are in order on (S, S_0), they do not form the vertices of a convex polygon. In fact P_2 is an interior point of the triangle $P_1 P_3 P_4$.

If $\Omega, P_1, P_2, \Omega'$ be in positive order on a hyperbolic branch S and $\Omega', P_3, P_4, \Omega$ be in negative order on S_0, Ω and Ω' being the points at infinity on S_0 which correspond to Ω and Ω' on S, respectively, then P_1, P_2, P_3, P_4 will be defined to be in order on (S, S_0). In this case P_1, P_2, P_3, P_4 are not *successive* vertices of a convex polygon. If P_1, P_2, P_3, P_4 be in positive order on (S, S_0), they are in negative order on (S_0, S).

It will at once appear that the above four points P_1, P_2, P_3, P_4 which are in order on (S, S_0) cannot be in order on a convex oval, ellipse, parabola or single branch of a hyperbola.

Lemma I

If P_1, P_2, P_3, P_4, P_5 be any five intersections of a hyperbola with a convex oval V, then they must lie on the same branch of the hyperbola.

If not, at least three, P_1, P_2, P_3, will lie on one branch S and at least one, P_4, on the obverse branch S_0.

First suppose P_1, P_2, P_3 are distinct and are in order on S, with S_0 between P_3 and P_1. Then P_2 is an interior point of the triangle $P_1 P_3 P_4$ and consequently P_1, P_2, P_3, P_4 cannot lie on a convex polygon.

If P_1 and P_2 form a two-point P on V then S and V will have a common tangent t at P. P_3 and P_4 as belonging to (S, S_0) will be on opposite sides of t and as belonging to V will lie on the same side of t, which is impossible.

If P_1, P_2, P_3 form a three-point P on V then S crosses V at P and consequently meets V again at some point P′ different from P. The argument of the last case will hold again.

It should be borne in mind that a conic can meet V at P either in a one-point or a two-point or at most a three-point and that the total number of points thus counted at which an ellipse, parabola or single branch of a hyperbola can intersect V is always *even*.

Lemma II.

If P_1, P_2, P_3......P_n $(n \geq 5)$ are any n distinct intersections of a conic S with a convex oval V, and if P_1, P_2, P_3......P_n are in order on V they are also in order on S.

If S be a hyperbola all the points lie on the same branch of the hyperbola, by Lemma I. The rest follows from definitions (*i*) and (*v*).

Cor. (*i*). These n points determine a unique positive sense on S as well as on V.

If Q be a point of S and T of V between two of the intersections P_r and P_s such that P_r, Q, P_s and P_r, T, P_s are in positive order on S and V, respectively, then evidently Q and T will fall on the same side (left) of $P_r P_s$.

If S be a hyperbolic branch and the obverse S_0 of S lie between P_r and P_s and if Q be taken on S_0 then Q will be

supposed to lie on the left of P_r P_s although it actually lies on the right. If S $(P_r\ P_s)$ denote the part of S constituted by all points Q and V $(P_r\ P_s)$ denote the part of V constituted by all points T and if S $(P_r\ P_s)$ and V $(P_r\ P_s)$ have no point common between P_r and P_s, then S $(P_r\ P_s)$ will lie entirely over or entirely under $V(P_r P_s)$. If S be a hyperbolic branch and if S_* lie between P_r and P_s, then $S(P_r\ P_s)$ will *include* S_*.

Cor. (*ii*). If P_1, P_2, P_3, P_4 are four distinct intersections of S and S', each of which is an ellipse, parabola or a single branch of a hyperbola, then P_1, P_2, P_3, P_4 form vertices of a convex polygon and if they are in order on S, they are also in order on S'. Corresponding to the positive sense along S there is a unique positive sense along S'.

$S(P_1\ P_2)$, $S(P_2\ P_3)$, $S(P_3\ P_4)$, $S(P_4\ P_1)$ will lie alternately under and over or over and under $S'(P_1\ P_2)$, $S'(P_2\ P_3)$, $S'(P_3\ P_4)$, $S'(P_4\ P_1)$, respectively.

LEMMA III.

If a hyperbolic branch S intersect a conic X at four points O_1, O_2, O_3, O_4, which are in order on both, and if S_*, the obverse of S, lies between O_4 and O_1, and X pass *over* S between O_4 and O_1, then X must be a hyperbola with one branch S_1 containing O_1, O_2, O_3, O_4 and the other branch S'_* falling between O_4 and O_1. Further, the eccentricity of X will be greater than that of S.

First suppose O_1, O_2, O_3, O_4 are all distinct. Since they are in order on S they are the successive vertices of a convex polygon and consequently cannot lie on two different branches of a hyperbola (*Sec.* 4). As the obverse of S lies between O_4 and O_1 and X passes over S between O_4 and O_1, X must be hyperbola with one branch S' containing O_1, O_2, O_3, O_4 and the other branch S'_* falling between O_4 and O_1.

If now from the mid-point O of the chord P_4P_1 lines be drawn to W_1 and W_2 as also to W'_1 and W'_2 where W_1 and W_2 are points at infinity on S and W'_1 and W'_2 are points at infinity on S', then evidently the angle W_1OW_2 falls within the angle $W'_1OW'_2$ and consequently the eccentricity of S' is greater than that of S, as the eccentricity increases with the asymptotic angle.

The cases where all the points O_1, O_2, O_3, O_4 are not distinct are treated similarly.

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Def. (*vi*).—An ordered range $R_n \equiv P_1, P_2, \ldots\ldots P_n$ of $n(\geq 5)$ points of intersection of a definite conic S with the oval V, will be called an *associated* range of *index n*. The conic S will be called the *associative* of R.

Each point of R_n where S crosses V, but does not touch, is to be counted as one point of R_n and each point where S over or under touches V is to be counted in general as two points of R_n, in determining the index n. Each point of R_n where S crosses V as well as touches V is to be counted in general as theree points of R_n. Besides the n points thus counted there may be other points of intersection of S with V between P_1 and P_n both inclusive. Such points when they exist will be called *extra points* of R_n. Extra points may exist between two distinct points of R_n. They may also fall in with any of the n associated points of R_n. If S over- or under-touches V at any point P_r of R_n we *may* count P_r as only one point of R_n, the other point at P_r counting among the extra points. Similarly if S cross-touches V at P_r we *may* count only one or two points at P_r as belonging to R_n, the rest counting among the extra points.

Def. (*vii*).—An associated range R_n will be called *regular* if the range do not possess any extra points. A regular range R will be denoted by (R) or (R_n).

Def. (*viii*).—If P_r and P_s be two distinct points of an associatde range R_n, the part $S(P_rP_s)$ of S, the associative of R_n, measured from P_r to P_s, in the positive sense, will be called the *curl* of R_n from P_r to P_s $(s > r)$. If S be a hyperbolic branch and if the obverse S_o of S lie between P_r and P_s then the curl $S(P_rP_s)$ will include S_o. The part of the conic S which is complementary to $S(P_rP_s)$ will be denoted by $S(P_sP_r)$.

Def. (*ix*).—If P_r and P_{r+1} be two distinct consecutive elements of a regular range (R_n), then the curl $S(P_rP_{r+1})$ will be called a regular curl. It will fall either entirely outside or entirely inside V. In the former case it will be called an *over curl* and in the latter case an *under curl*. If P_r and P_{r+1} coincide the over curl reduces to an *under point* and the under curl to an *over point*. If P_r, P_{r+1}, P_{r+2} coincide in an associated three-point, an over point and an under point coincide. The resulting point may be called an *over-under* point or an *under-over* point according as t tet ahi

limit of an under followed by an over curl or of an over followed by an under curl.

Def. (*x*).—The arc $V(P_r P_{r+1})$ of the oval V between two successive points P_r and P_{r+1} of the range (R_n) will be called the *r*-th *lap* of (R_n). The largest of the *n* laps $l'_n, l''_n, l'''_n, \ldots, l^n_n$ of (R_n) will be called the maximum lap of (R_n), and denoted by l_n. The sum of the *n* laps of (R_n) will be called the full lap of (R_n), or simply the lap of (R_n).

Def. (*xi*).—If (R_n) be of an even index *n*, the extreme curls, when they are both finite, will be both over or both under. In the former case (R_n) will be called an *under-range* and in the latter case an *over-range*. In the former case a point of either extreme laps will be an interior point of S, the associative of (R_n), and in the latter case a point of either extreme laps will be an exterior point of S. If there be a two-point of (R_n) at either extremity it will be an over-point for an over-range and an under-point for an under-range. If there be a three-point of (R_n) at either extremity it will be an over-under point for an over-range and an under-over point for an under-range. An over-range will be said to belong to a *first category* and an under-range to a *second category*.

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Def. (*xii*).—A regular range of index 6 will be called a *hexadic range* or simply a *hexad*. A hexad is either an over-hexad or an under-hexad. A hexad will be denoted simply by R.

An element of a hexad may be either a one-point or a two-point or a three-point. If a hexad consist of two three-points the associative of the hexad is not fully determined by them as each three-point contains only two definite points. A fifth definite point must therefore exist elsewhere to define the associative.

Def. (*xiii*).—If $R \equiv P_1, P_2, P_3, P_4, P_5, P_6$ be a hexad then the mid-points $P'_1, P'_2, P'_3, P'_4, P'_5$ of the five successive laps of R will be called the *mean-points* of R or simply the *means* of R. The conic S′ through the five means of R will be called the *mean-associative* of R.

It may be observed that the five means of R are *in every case* five definite points of V and consequently suffice to define S′. If R do not contain a three-point all the five means are distinct. If R contain a three-point two of the means coincide forming a definite two-point on V.

LEMMA IV.

The associative S of a hexad R meets the mean associative S′ of R at four points O_1, O_2, O_3, O_4, in order on S and S′, lying on S between P_1 and P_6, both inclusive, and on S′ between P'_1 and P'_5, both inclusive.

First suppose R consists of six one-points P_1, P_2, P_3, P_4, P_5, P_6. Then P'_1 and P'_2 will lie on opposite sides of S and consequently the curl $S'(P'_1\ P'_2)$ must meet S at some point O_1 of S′ between P'_1 and P'_2. Similarly $S'(P'_2\ P'_3)$, $S'(P'_3\ P'_4)$ and $S'(P'_4\ P'_5)$ will meet S at O_2, O_3 and O_4, respectively, such that O_2 lies between P'_2 and P'_3, O_3 lies between P'_3 and P'_4 and O_4 lies between P'_4 and P'_5. Thus O_1, O_2, O_3, O_4 are in order on S′ between P'_1 and P'_5. Consequently they are also in order on S (Lemma II, Cor. *ii*).

Again if $S'(P'_1\ P'_2)$ be a regular curl, it will be on the same side of V as one of the regular curls $S(P_1\ P_2)$ and $S(P_2\ P_3)$ and therefore will cross $S(P'_1\ P'_2)$ at O_1, either between P_1 and P_2 or between P_2 and P_3. If $S'(P'_1\ P'_2)$ be not regular, it will split up into two or more regular curls one of which will cross $S(P_1\ P_3)$ at O_1 between P_1 and P_3. Similarly O_4 will lie on S between P_4 and P_6.

If the hexad R has a two-point at P on V, S′ will pass through P. One of the points O will therefore be at P. If the hexad R has a three-point at P on V, S′ will touch V at P. Two of the points O will therefore be at P. If P_1 be a two-point or a three-point O_1, P_1, P'_1 coincide and if P_6 be a two-point or a three-point O_4, P_6, P'_5 coincide.

Def. (*xiv*).—A hexad R′ each of whose extreme elements fall between the extreme elements P_1 and P_6 of R, or coincide with either, and whose associative S′ is the mean associative of R, will be called an *inner mean derived* of R.

LEMMA V.

To every hexad of a given category there exists at least one inner mean derived of the same category.

Suppose R is an over-hexad with six distinct elements P_1, P_2, P_3, P_4, P_5, P_6. Then P'_1, P'_2, P'_3, P'_4, P'_5, the five means of

R, are all distinct and P'_1 and P'_5 lie outside S. There exists four intersections O_1, O_2, O_3, O_4 of S′ with S which lie between P'_1 and P'_5 on S′ and between P_1 and P_6 on S (Lemma IV). Consequently $S'(P'_5 P'_1)$, the complementary of $S'(P'_1 P'_5)$, will fall entirely outside S, and S′ will have no point on $S(P_6 P_1)$.

The regular range (R'_s) of intersections of S′ with V, between P_1 and P_6, must be of an even index, as it consists of all the intersections of S′ with the closed figure consisting of $V(P_1 P_6)$ and $S(P_6 P_1)$. Consequently (R'_r) is either an over-range or an under-range.

(R'_s) must be an over-range, for if P'_s be the upper extreme element of (R'_s) then it must obviously lie in $V(P_5 P_6)$. If (R'_s) were an under-range then $S'(P'_s P'_1)$ would enter V at P'_s and as it could not meet $V(P_5 P_6)$ again, would cross the curl $S(P_5 P_6)$ at some point, which is impossible, as $S'(P'_s P'_1)$ being a part of $S'(P'_5 P'_1)$ lies entirely outside S.

If R be an under-hexad we can similarly shew that (R'_s) will be an under-range.

The cases where R has one or more two-points or three-points do not present any special difficulties and can be treated in a similar way.

It is of interest to note that although in general a two-point of R gives a one-point of (R'_s) and a three-point of R gives a two-point of (R'_s) these one-points and two-points may become two-points or three-points by association in (R'_s).

If in (R'_s) the index is 6 then (R'_s) is itself a hexad of the same category as R. If the index be 8 then (R'_s) gives two hexads P'_1, P'_2, P'_3, P'_4, P'_5, P'_6 and P'_3, P'_4, P'_5, P'_6, P'_7 P'_8 of the same category as R and one hexad P'_2, P'_3, P'_4, P'_5, P'_6, P'_7 of the opposite category.

It may be observed that in (R'_s) the first six elements always constitute a hexad of the same category as R. This hexad may be called the *leader* of (R'_s) or the leading inner mean derivate of R.

Def. (*xv*).—If R, R′, R″,..., $R^{(n)}$,...be a sequence of hexads such that each hexad after R is an inner mean derivate of the one which immediately precedes it, then R, R′, R″,...$R^{(n)}$...will be called an *inner mean derived sequence of hexads.*

LEMMA VI.

If $l,\ l',\ l'',\ldots l^{(n)}\ldots$ be the maximum laps of an inner mean derived sequence of hexads R, R', R'',...$R^{(n)}$,... respectively, then $l,\ l',\ l'',\ldots l^{(n)}$ ....will form a monotone sequence of zero limit.

It is easily seen that if $l_r^{(n)}$ be any lap of $R^{(n)}$ it must lie either (*i*) in a certain semi-lap $\frac{1}{2}\,l_s^{(n-1)}$ of $R^{(n-1)}$ or (*ii*) in two consecutive semi-laps $\frac{1}{2}\,l_s^{(n-1)}$ and $\frac{1}{2}\,l_{s+1}^{(n-1)}$ of $R^{(n-1)}$.

In the first case we have $l_r^{(n)} \lessapprox \frac{1}{2}\,l_s^{(n-1)}$.

In the second case we have $l_r^{(n)} \lessapprox \frac{1}{2}(l_s^{(n-1)} + l_{s+1}^{(n-1)})$.

Again as the six points of $R^{(n)}$ cannot all lie in different laps of $R^{(n-1)}$, of which the number is only five, there must exist at least one lap of $R^{(n-1)}$ for which case (*i*) holds.

We have from case (*i*) $l^{(n)} \lessapprox \frac{1}{2}l_s^{(n-1)} \lessapprox \frac{1}{2}l^{(n-1)}$ and from case (*ii*) $l^{(n)} \lessapprox \frac{1}{2}\ (l_s^{(n-1)} + l_{s+1}^{(n-1)}) \lessapprox l^{(n-1)}$.

In either case $l^{(n)} \lessapprox l^{(n-1)}$ and consequently $[l^{(n)}]$ is a monotone decreasing sequence. It must therefore have a limit L, which is either zero or finite. We shall shew that L cannot be finite.

If L be finite then a value m of n exists such that $l^{(n)} - L \angle \epsilon$ where ϵ is an arbitrary given length, for all values of $n \geq m$. We may take $\epsilon = L/100$.

Consequently $l^{(m)} - L,\ l^{(m+1)} - L,\ l^{(m+2)} - L,\ l^{(m+3)} - L,\ l^{(m+4)} - L,\ l^{(m+5)} - L$, are each less than ϵ.

We have either $l^{(m+5)} \lessapprox \frac{1}{2}l_r^{(m+4)}$, case (*i*), or $l^{(m+5)} \lessapprox \frac{1}{2}(l_r^{(m+4)} + l_{r+1}^{(m+4)})$, case (*ii*).

In the former case put $l^{(m+5)} = L + \epsilon_5$ and $l_r^{(m+4)} = l^{(m+4)} - \epsilon_4^{(r)} = L + \epsilon_4 - \epsilon_4^{(r)}$, where $0 \leqq \epsilon_5 \leqq \epsilon_4 < \epsilon$. Therefore $L + \epsilon_5 \leqq \frac{1}{2}(L + \epsilon_4 - \epsilon_4^{(r)})$ or $L + 2\epsilon_5 + \epsilon_4^{(r)} \leqq \epsilon_4$ or $L < \epsilon$, which is absurd.

In the latter case put $l^{(m+5)} = L + \epsilon_5$, $l_r^{(m+4)} = l^{(m+4)} - \epsilon_4^{(r)} = L + \epsilon_4 - \epsilon_4^{(r)}$ and $l_{r+1}^{(m+4)} = l^{(m+4)} - \epsilon_4^{(r+1)} = L + \epsilon_4 - \epsilon_4^{(r+1)}$

where $0 \leqq \epsilon_5 \leqq \epsilon_4 \leqq \epsilon$, $0 \leqq \epsilon_4^{(r)}$ and $0 \leqq \epsilon_4^{(r+1)}$.

Therefore $L + \epsilon_5 \leqq \frac{1}{2}(2L + 2\epsilon_4 - \epsilon_4^{(r)} - \epsilon_4^{(r+1)})$, or $\epsilon_4^{(r)} + \epsilon_4^{(r+1)} \leqq 2(\epsilon_4 - \epsilon_5) \leqq 2\epsilon$. Therefore $\epsilon_4^{(r)}$ and $\epsilon_4^{(r+1)}$ are each less than 2ϵ.

Again we have either $l_r^{(m+4)} \leqq \frac{1}{2} l_s^{(m+3)}$, case (*i*), or $l_r^{(m+4)} \leqq \frac{1}{2}(l_s^{(m+3)} + l_{s+1}^{(m+3)})$, case (*ii*).

In the former case put $l_r^{(m+4)} = L + \epsilon_4 - \epsilon_4^{(r)}$ and $l_s^{(m+3)} = l^{(m+3)} - \epsilon_3^{(s)} = L + \epsilon_3 - \epsilon_3^{(s)}$

where $0 \leqq \epsilon_5 \leqq \epsilon_4 \leqq \epsilon_3 < \epsilon$ and $0 \leqq \epsilon_3^{(s)}$.

We obtain $L + \epsilon_4 - \epsilon_4^{(r)} \leqq \frac{1}{2}(L + \epsilon_3 - \epsilon_3^{(s)})$ or $L + \epsilon_3^{(s)} \leqq \epsilon_3 + 2\epsilon_4^{(r)} - 2\epsilon_4 \leqq (\epsilon_3 - \epsilon_4) - \epsilon_4 + 2\epsilon_4^{(r)}$ or $L < 5\epsilon$, which is absurd.

In the latter case put $l_r^{(m+4)} = L + \epsilon_4 - \epsilon_4^{(r)}$, $l_s^{(m+3)} = L + \epsilon_3 - \epsilon_3^{(s)}$, $l_{s+1}^{(m+3)} = L + \epsilon_3 - \epsilon_3^{(s+1)}$ where $0 \leqq \epsilon_4 \leqq \epsilon_3 < \epsilon$ and $\epsilon_3^{(s)}$ and $\epsilon_3^{(s+1)}$ are each $\geqq 0$.

Whence we obtain $\epsilon_3^{(s)} + \epsilon_3^{(s+1)} \leqq 2(\epsilon_3 - \epsilon_4) + 2\epsilon_4^{(r)} < 6\epsilon$.

Therefore $\epsilon_3^{(s)}$ and $\epsilon_3^{(s+1)}$ are each less than 6ϵ.

Similarly we have either $l_{r+1}^{(m+4)} \leqq \frac{1}{2}(l_t^{(m+3)})$, case (*i*),

or $l_{r+1}^{(m+4)} \leqq \frac{1}{2}(l_t^{(m+3)} + l_{t+1}^{(m+3)})$, case (*ii*).

The former leads to absurdity and the latter gives $\epsilon_3^{(t)}$ and $\epsilon_3^{(t+1)}$ each less than 6ϵ,

where $$l_t^{(m+3)} = l^{(m+3)} - \epsilon_3^{(t)} \text{ and } l_{t+1}^{(m+3)} = l^{(m+3)} - \epsilon_3^{(t+1)}.$$

Now $l_s^{(m+3)}$ and $l_{s+1}^{(m+3)}$ cannot be identical with $l_t^{(m+3)}$ and $l_{t+1}^{(m+3)}$, respectively, for a little consideration shews that from two consecutive laps of $R^{(m+3)}$ we can derive at most one lap of $R^{(m+4)}$ which falls under case (*ii*).

Again $l_s^{(m+3)}$, $l_{s+1}^{(m+3)}$, $l_t^{(m+3)}$, $l_{t+1}^{(m+3)}$ cannot be all different, A little consideration shews that two consecutive laps of $R^{(m+4)}$ can at most be derived from three consecutive laps of $R^{(m+3)}$ if either laps of $R^{(m+4)}$ fall under case (*ii*).

We conclude therefore that $l_{s+1}^{(m+3)}$ is identical with $l_t^{(m+3)}$. We have consequently three consecutive laps of $R^{(m+3)}$, *viz.*, $l_s^{(m+3)}$ $l_{s+1}^{(m+3)}$, $l_{s+2}^{(m+3)}$ from which the two consecutive laps $l_r^{(m+4)}$ and $l_{r+1}^{(m+4)}$ of $R^{(m+4)}$ are derived. We have at the same time $\epsilon_3^{(s)}$, $\epsilon_3^{(s+1)}$, $\epsilon_3^{(s+2)}$ each less than 6ϵ.

By continuing the same reasoning we get in $R^{(m+2)}$ four consecutive laps $l_u^{(m+2)}$, $l_{u+1}^{(m+2)}$, $l_{u+2}^{(m+2)}$, $l_{u+3}^{(m+2)}$ such that $\epsilon_2^{(u)}$, $\epsilon_2^{(u+1)}$ $\epsilon_2^{(u+2)}$ $\epsilon_2^{(u+3)}$, $\epsilon_2^{(u+3)}$ are each less than 14ϵ, and in $R^{(m+1)}$ five consecutive laps $l_v^{(m+1)}$, $l_{v+1}^{(m+1)}$, $l_{v+2}^{(m+1)}$, $l_{v+3}^{(m+1)}$, $l_{v+4}^{(m+1)}$ such

that $\epsilon_2^{(v)}, \epsilon_2^{(v+1)}, \epsilon_2^{(v+2)}, \epsilon_2^{(v+3)}, \epsilon_2^{(v+4)}$ are each less than 30ϵ.

But as $R^{(m+1)}$ possesses only five laps we have $v=1$.

Now $R^{(m+1)}$ is an inner mean derived of $R^{(m)}$ and therefore must possess at least one lap for which case (*i*) holds. This leads to $L \angle 61\,\epsilon$, which is absurd.

Hence we conclude that the sequence $[1^{(n)}]$ has zero limit.

Cor. (*i*). If $\lambda, \lambda', \lambda'', \ldots \lambda^{(n)}$ be the entire laps of $R, R', R'', \ldots\ldots R^{(n)}$ respectively, then the limit of the sequence $[\lambda^{(n)}]$ is zero.

Cor. (*ii*). There is a unique point common to all the laps of the sequence $[\lambda^{(n)}]$.

This unique point will be called a *hexadic point* of V, defined by the inner mean derived sequence of hexads, $R, R', R'', \ldots\ldots$

Cor. (*iii*). Every elementary oval possesses some hexadic points.

Take any five distinct points of V. The associative S of these five points will meet V in at least another point. Consequently there exists at least six hexads on V of which the associative is S. Any of these six hexads with a sequence of inner mean derivates defines a hexadic point. Suppose K is a hexadic point thus defined. Now every five-pointic conic of V cannot pass through K for then V would be a conic. Hence a hexadic range exists on V whose associative does not pass through K, and whose laps do not contain K. A sequence of mean derivates of this hexad will define another hexadic point.

7

We will now proceed to the proof of Böhmer's Theorem (B).

If every hexadic point of an elementary convex oval be elliptic, the conic through any five definite points of V will be an ellipse.

If possible, suppose a definite five-pointic conic of V exists which is a hyperbola (S, S_0). Suppose $(R_n)=P_1, P_2, P_3, \ldots\ldots P_n$ denotes the complete regular range of intersections of (S, S_0) with V, where initial point P_1 may be any one of the intersections. The index of the range (R_n) must be an even number $\leqq 6$, and all the points of the range will lie on the same branch S of the hyperbola (Lemma I).

The points at infinity Ω and Ω' on S are outside V. Suppose Ω, P_1, Ω' on S are in order and there is no point of (R_n) between Ω and P_1. Then evidently there will be no point of (R_n) between P_n and Ω'. The range (R_n) will therefore be over-hexadic.

Consider the leader $R = P_1, P_2, P_3, P_4, P_5, P_6$ of (R_n) which will be also over-hexadic and a sequence $R', R'', \ldots R^{(n)}, \ldots$ of successive inner mean derivates of R of the same category as R. Suppose $S', S'', \ldots, S^{(n)}, \ldots$ are the associatives of $R', R'', \ldots R^{(n)}, \ldots$ respectively.

S and S′ have four intersections O_1, O_2, O_3, O_4 which lie in order on S between P_1 and P_6 and on S′ between P'_1 and P'_5 (Lemma IV), and P'_1 and P'_5 are outside S. Consequently the range O_1, O_2, O_3, O_4 made by S′ on S is an over-range, that is, S′ goes over S between O_4 and O_1. Hence S′ is a hyperbolic branch which has its obverse between P'_1 and P'_5 and has its eccentricity greater than that of S (Lemma III).

Suppose $P(s)$ is the hexadic point defined by the sequence $R, R', R'', \ldots$ Then for each neighbourhood $(s-\delta, s+\delta)$ of $P(s)$ there exists a value m of n such that $R^{(n)}$ lies in this neighbourhood, for $n \geqq m$.

But every hexadic point of the oval is elliptic and consequently a δ exists such that the conic through every five definite points of the neighbourhood $(s-\delta, s+\delta)$ of $P(s)$ is an ellipse, which is contradicted by our conclusion that hexads exist in this neighbourhood for which the associative conic is a hyperbola. Thus Böhmer's theorem (B) is completely proved.

It is worthy of note that although the associative $S^{(n)}$ of hexad $R^{(n)}$ consisting of two three-points may not be considered as passing through five definite points in the lap of $R^{(n)}$, the mean associative $S^{(n+1)}$ of $R^{(n)}$ always pass through five definite points in this lap. If the regular range in which $S^{(n+1)}$ meets V in the lap of $R^{(n)}$ be denoted by $(R^{(n+1)})$, then $(R^{(n+1)})$ will consist of no less than six points, at least five of which are always definite. The hexadic point $P(s)$ which has been defined by the sequence $[R^{(n)}]$ may therefore be equally well defined by the sequence $[(R^{(n+1)})]$.

GENERALISATION OF CERTAIN THEOREMS IN THE HYPERBOLIC GEOMETRY OF THE TRIANGLE.*

By

S. Mukhopadhyaya and G. Bhar.† (1919.)

Introduction.

The geometry of the triangle on the hyperbolic plane has many remarkable features which are absent in the geometry of the plane triangle and which are brought out the more prominently by a purely geometrical treatment. We will consider two well-known theorems in the geometry of the hyperbolic triangle with a view to elegant geometrical demonstrations and extensions to the case where one or more of the vertices are ideal or improper points. In the course of the investigations we will come to some very remarkable new theorems.

We have in Euclidean geometry the two well-known theorems: (*i*) The three internal bisectors of the angles of a triangle or two external and one internal bisector meet at a point. (*ii*) The three perpendiculars on the sides of a triangle from the opposite vertices meet at a point.

We will discuss their analogues on the hyperbolic plane with actual, ideal or improper vertices.

A system of lines on a hyperbolic plane are said to meet at an ideal point when they are all perpendicular to the same straight line. This straight line is uniquely representative of the ideal point. The system of lines are said to meet at an improper point when they are parallel to one another in the same sense.

Theorem I:—*The three internal bisectors of the angles of a hyperbolic triangle ABC meet at an actual point.*

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† This paper was read before the Calcutta Mathematical Society in an abstract form. I owe to my pupil Mr. G. Bhar, M.Sc., the present expanded form of my paper embracing all the different cases and the carefully drawn diagrams.—S. M.

The internal bisector of an angle A must meet the opposite side at some point D. The internal bisector of B will meet AD at some point O. The perpendiculars from O on AC and BC are each equal to the perpendicular from O on AB. Therefore the internal bisector of the angle C passes through O.

Theorem II :—*The external bisectors of any two angles B and C of a hyperbolic triangle ABC meet the internal bisector of the third angle A at an actual, ideal or improper point.*

If any two of the three bisectors pass through an actual point the third can be shown to pass through the same actual point as in *Theorem I.*

If no two of the three bisectors meet at an actual point, then the two external bisectors of the angles B and C either meet at an ideal point or are parallel.

Suppose the two external bisectors BD and CE meet at an ideal point, that is, have a common perpendicular DE (fig. 1). Then it is easily shown that D and E lie on the side of BC away from A, for otherwise it would follow that the sum of four angles of a hyperbolic quadrilateral are together greater than four right angles or that an exterior angle of a triangle is less than the interior opposite angle.

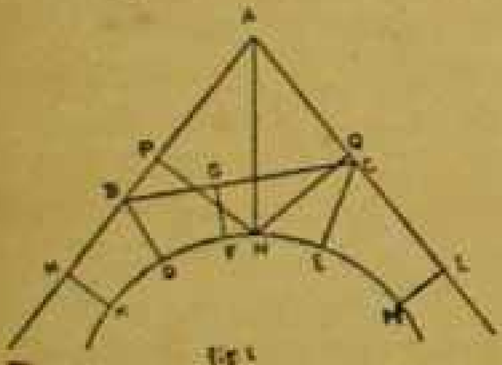


fig 1

This common perpendicular DE cannot meet BC produced either towards B or towards C, for in either case an exterior angle would be less than an interior opposite angle. Neither can DE be parallel to BC either towards B or towards C, for then an angle of parallelism would be greater than a right angle. Therefore DE and BC meet at an ideal point, that is, have a common perpendicular GF, where it is easy to see that G lies on BC between B and C and F lies on DE between D and E.

Produce AB to H and ED to K making BH=BG and DK=DF. Also produce AC to L and DE to M making CL=CG and EM=EF. Then HK is a common perpendicular to AB and ED and LM is a common perpendicular to AC and ED. Also HK=GF=LM.

Bisect KM at N. Then the perpendiculars NP and NQ on AB and AC are equal from the equality of the quadrilaterals NPHK and

NQLM. Therefore AN is the internal bisector of the angle A. It is also evidently perpendicular to DE. Therefore BD, CE and AN have a common perpendicular and therefore meet at an ideal point.

If the two external bisectors of the angles B and C are parallel, the internal bisector of the angle A cannot meet either as then the three would pass through a common actual point. The internal bisector therefore passes between the two parallel external bisectors without meeting either and therefore must be parallel to both in the same sense.

Corollary to Theorem II:—In the triangle ABC if g be the foot of the perpendicular on BC from the point O, the actual point of concurrence of the internal bisectors of the triangle ABC and G, the foot of the perpendicular on BC from O′ the actual, ideal or improper point of concurrence of the internal bisector of the angle A and the external bisectors of the angles B and C, then $Bg = CG$.

For, $$AB - Bg = AC - Cg,$$

also $$AB + BG = AC + CG \text{ and } Bg + Cg = CG + BG$$

as is evident from constructions of Theorems I and II when O′ is an actual or ideal point. When O′ is an improper point similar constructions have to be made.

Theorem III:—*The three perpendiculars from the vertices of a triangle in the hyperbolic plane on the opposite sides meet at a point, actual, ideal or improper.*

Let ABC be the given triangle and AD, BE, CF the three perpendiculars from A, B, C on the opposite sides. Draw α, β, γ through A, B, C at right angles to AD, BE, CF respectively.

Case 1. Suppose β and γ meet at an actual point. Then it will be shown that α, β and α, γ will also meet at actual points.

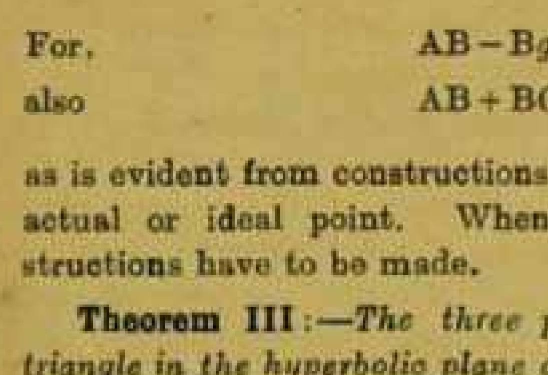


fig 2.

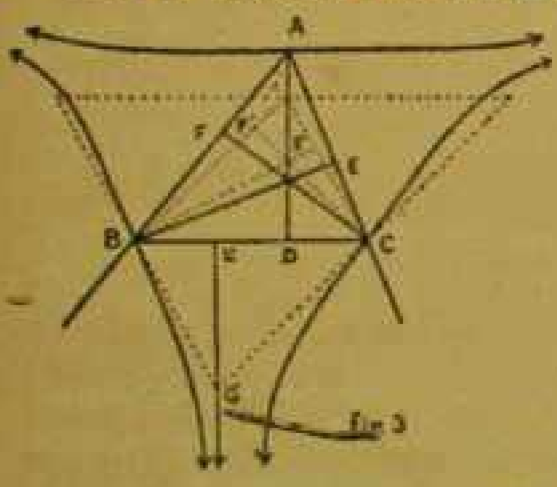
Let G be the point of intersection of β and γ (fig. 2). Produce GC to H and GB to K making CH=GC and BK=GB. Then the join of HK will pass through A and will be perpendicular to AD.

From G, H, K draw perpendiculars GL, GM, GN, HO, HP, HQ,

KR, KS, KT on the sides BC, CA and AB of the triangle ABC. Then because GC=HC and CF is the common perpendicular to HG and AB we have GN=HQ. Again from the congruent triangles GBN and KBT we get GN=KT. It follows therefore HQ=KT and similarly HP=KS. If H and K be joined, the line HK will pass through A, for otherwise it will cut BA and CA or each side produced through A in two points each of which shall be the middle point of the segment HK which is absurd. Now HO=GL=KR and from the congruent quadrilaterals HODA and KRDA, it is clear that angle DAH is equal to angle DAK ; thus AD is perpendicular to HK. Hence the perpendiculars from the vertices A, B, C are the perpendicular bisectors of the sides of the triangle GHK and they therefore meet at a point (Theorem of Bolyai).

It is important to observe that as BC=$\frac{1}{2}$ OR=OD, we get BD=OC=CL.

Case 2. Suppose now that β and γ are parallel, that is, meet at an improper point (fig. 3). If on AD, between A and D, a point A′ be taken, perpendiculars BE′ and CF′ from B and C on the sides CA′ and BA′ of the triangle A′BC will lie on the sides of BE and CF away from BC. Therefore if β' and γ' be drawn through B and C perpendiculars to BE′ and CF′, they meet at an actual point G′



and it can be proved, as in *Case 1*, that α', β' and α', γ' also meet in actual points K′ and H′ where α' is the perpendicular through A′ to A′D. If G′L′ be drawn from G′ perpendicular to BC, then because DC=BL′, as A′ moves along A′D towards A, G′ will move along L′G′ away from BC and finally when A′ coincides with A, G′ moves off to infinity so that β' and γ' coincide with β and γ. Now as BG′ and CG′ are always equal to BK′ and CH′, respectively, as G goes to infinity, H′ and K′ at the same time go to infinity. Again as the theorem is true in all particular cases it is also true in the limiting case; α' which is always perpendicular to A′D will remain so when A′ moves to A, that is, when α' coincides with α. Thus α which is perpendicular to AD meets β and γ at improper points.

Case 3. Let now β and γ be non-intersecting lines, that is, let them meet at an ideal point. They will have a common perpendicular GG' representative of that point.

We may suppose the angles ABC and ACB to be acute, for at least two of the angles of a triangle must be so. Then angles CBG and BCG' are both acute, consequently GG' cannot cut BC.

GG' cannot also cut AB and AC ; for supposing GG' cuts AB and AC (fig. 4) at the points G_3 and G_2, on GB and $G'C$ produced through B and C we can take two points H and K such that $GB=BH$ and $G'C=CK$. Perpendiculars erected at H and K to HB and KC will meet BA and CA, produced if necessary at H_3, H_2 and K_3, K_2 respectively. Further these perpendiculars must cut each other at a point Z. It can now be easily shown that the triangles ZH_2K_2 and ZH_3K_3 are both isosceles, so that the bisector of the angle H_2ZK_2, which is also the bisector of the angle H_3ZK_3, must be perpendicular to both the intersecting lines H_2K_2 and H_3K_3 which is absurd.

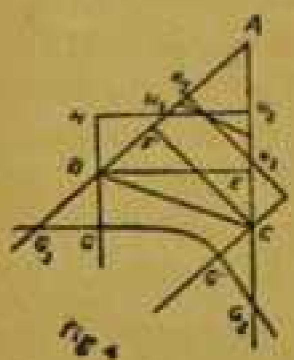


Fig. 4

Again it is not possible that GG' shall cut one of the sides AB and AC and be parallel to the other. Supposing that GG' cuts AB at G_3 and is parallel to AC, if on GB produced we take as before a point H such that $GB=BH$ and erect a perpendicular at H, this perpendicular will cut BA, produced if necessary, at a point H_3, for the triangles G_3GB and H_3HB are congruent. Consequently it will cut CA, produced if necessary, at a point H_2. Now as $GB=BH$ and BE is the common perpendicular to HG and AC and also GG' is parallel to AC, the perpendicular through H to HG must also be parallel to CA. Thus this perpendicular cuts CA and is at the same time parallel to CA which is absurd.

In an exactly similar way it can be shown that it is not possible that GG' shall cut one of the sides AB and AC and be non-intersecting to the other.

Further it is easy to see in like manner that GG' cannot be parallel to both AB and AC.

GG' must therefore be non-intersecting to both AB and AC.

Let G_1L, G_2M, G_3N be the common perpendiculars between the line GG' and BC, CA and BA respectively (fig. 5). If GB and $G'C$ be produced through B and C to H and K making $GB=BH$ and $G'C=CK$ and perpendiculars HH' and KK' be erected at these points, these perpendiculars cannot cut either BC, CA or AB; for supposing that any of these perpendiculars cuts any of the sides BC, CA or AB, it can be shown from the properties of congruent figures that GG' must then also meet one of the sides BC, AC or AB which we have seen is not possible.

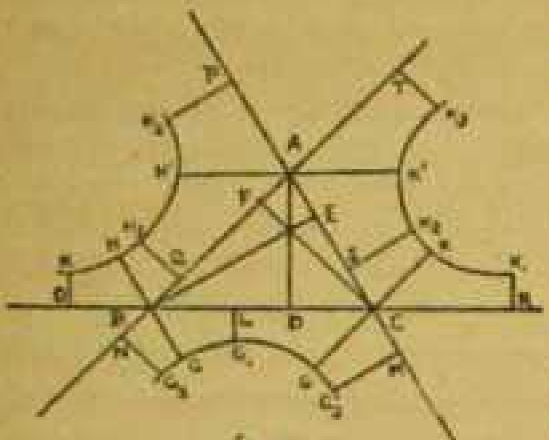


fig 5.

Let H_1O, H_2P, H_3Q, be the common perpendiculars between HH' and BC, CA and AB respectively and K_1R, K_2S, K_3T be the common perpendiculars between KK' and the same three lines respectively. Then it is easy to see from congruent figures that $H_1O=G_1L=K_1R$, $K_2S=G_2M=H_2P$ and $H_3Q=G_3N=K_3T$. If now $H'K'$ be the common perpendicular between HH' and KK', $H'K'$ must pass through A; for considering the two figures $APH_2H'H_3QA$ and $ASK_2K'K_3TA$ in which $H_3Q=K_3T$ and $H_2P=K_2S$, it can be shown that if $H'K'$ does not pass through A, it will cut AB and AC, produced if necessary through A, in two points each of which shall be the middle point of the finite segment $H'K'$ which is absurd. Further from the equality of the figures $H'ADOH_1$ and $K'ADRK_1$ it is clear that $\measuredangle H'AD=\measuredangle K'AD$ and $AH'=AK'$, so that AD is perpendicular to $H'K'$ through its middle point A.

Thus AD, BE and CF are the perpendicular bisectors of the common perpendiculars $H'K'$, HG and $G'K$ between HH', KK'; GG', HH' and GG', KK' respectively. And these will be proved to be concurrent later on. See *Theorem IV, Case II* (*i*).

Cor. to Theorem III:—$BD=CL$. This is evident from the constructions in the different cases.

From the above corollary and the constructions in the different cases of Theorem III, an important result can be deduced, as has been pointed out by my pupil Mr. R. C. Bose.

Suppose a, b, c, are the sides of the triangle ABC, and a_1, b_1, c_1 are the corresponding perpendiculars on them from the opposite vertices.

In *Case 1*, the acute angle in each of the three three-right-angled quadrilaterals ADRK, BEMG, CFQH of fig. 2, has opposite to it a pair of sides equal to (a, a_1), (b, b_1), (c, c_1) respectively. This acute angle is the same in all the three three-right-angled quadrilaterals being equal to half the sum of the angles of the triangle GHK.

In *Case 2*, (a, a_1), (b, b_1), (c, c_1) are three pairs of complementary segments.

In *Case 3*, fig. 5, there are three rectangular pentagons $ADRK_1K'$ $BEPH_2H$ and $CFNG_3G'$, which have a pair of adjacent sides equal to (a, a_1), (b, b_1) and (c, c_1), respectively. The sides opposite to the above pairs in the corresponding rectangular pentagons, are K_1K', $H_2H'_1$ G_3G', respectively, and each of these is equal to half the sum of KK', HH', GG'.

These results can be summed up in the following general theorem: Suppose XLX_1, YMY_1, ZNZ_1 are three right angles having the pairs of arms (LX, LX_1), (MY, MY_1), (NZ, NZ_1) equal to (a, a_1), (b, b_1), (c, c_1) respectively, and suppose x, x_1, y, y_1, z, z_1 are perpendiculars to LX, LX_1, MY, MY_1, NZ, NZ_1, at X, X_1, Y, Y_1, Z, Z_1 respectively. Then

If the pair (x, x_1) meet in an actual point at an angle α, each of the pairs (y, y_1), (z, z_1) will meet in an actual point at an angle α. If the pair (x, x_1) meet in an improper point, each of the pairs (y, y_1), (z, z_1) will meet in an improper point. If the pair (x, x_1) meet in an ideal point at a divergence δ, each of the pairs (y, y_1),— (z, z_1) will meet in an ideal point at divergence δ.

Mr. R. C. Bose has also pointed out that from the above theorem an elegant synthetic proof of the " difficult " median theorem of a triangle can be deduced.

A simpler and more elegant proof of *Theorem III* is given in *Theorem V*, which is the most general form of *Theorem III*. The present proof is of interest on account of its important *corollary*.

Definitions:—

The *symmetric* of two given directed lines is the locus of the middle points of all lines which are equally inclined to the two lines.

Every line perpendicular to the symmetric or passing through the symmetric, when the symmetric is a point, either meets the two given lines at equal angles or have equal common perpendiculars from them.

The symmetric of two given directed lines which meet at an actual point is the internal bisector of the angle between them.

The symmetric of two given directed lines which meet at an ideal point and have consequently a common perpendicular between them is the line bisecting this common perpendicular at right angles, if the given lines are directed in the same sense with respect to this common perpendicular; but if they are directed in opposite senses from this common perpendicular the symmetric reduces to the middle point of the common perpendicular, for it is evident that every line which is equally inclined to the two given directed lines passes through the middle point of the common perpendicular and is bisected at that point.

If the two given directed lines are parallel, the symmetric is a third parallel which is equidistant from them, provided the given lines are both directed in the same sense as, or opposite sense to, the direction of parallelism. Otherwise the symmetric will be defined to be the improper point to which the parallel lines converge.

With these definitions we proceed to prove the following comprehensive theorem.

Theorem IV :—*The symmetrics of any three co-planer lines, which are not concurrent, taken two and two in any three ways such that the same line has opposite senses in the two different pairs in which it occurs, are concurrent, the concurrency being understood as follows* :—

(*a*) if the three symmetrics are straight lines, they will meet at an actual, ideal or improper point ;

(*b*) if two of them be straight lines and the third a point, then the point will lie on the common perpendicular to the first two;

(*c*) if one of the symmetrics be a straight line and the other two points, then the straight line will be perpendicular to the join of the two points;

(*d*) if all the three symmetrics be points they will be collinear.

Let a, b, c represent any three coplanar lines which are not concurrent. If b and c meet at an actual point we will denote this point by α. If b and c meet at an ideal point, they have a common perpendicular. The ideal point or the common perpendicular may be indifferently denoted by α. If b and c meet at an improper point, then this improper point will be denoted by α. Similarly the points

of meeting, actual, ideal or improper, of the two lines c and a will be denoted by β and that of the lines a and b by γ.

The line a is directed in two ways and may be represented as such by $\beta\gamma$ and $\gamma\beta$. If β and γ be actual points this is obvious; if β and γ be ideal points then $\beta\gamma$ will represent line a as the common perpendicular between β and γ directed from β towards γ. Similarly if β be an actual point and γ an ideal point, then $\beta\gamma$ will represent the line a directed from β towards γ to which it is perpendicular. Similar interpretation may be given in every case.

The three lines a, b, c can be taken in groups of directed pairs, two and two, only in four ways satisfying the conditon that if any one of the lines a occur as $\beta\gamma$ in one pair it can only appear as $\gamma\beta$ in another pair. These groups are:

(1) $\beta\alpha,\ \gamma\alpha;\quad \gamma\beta,\ \alpha\beta;\quad \alpha\gamma,\ \beta\gamma;$

(2) $\beta\alpha,\ \gamma\alpha;\quad \beta\gamma,\ \alpha\beta;\quad \alpha\gamma,\ \gamma\beta;$

(3) $\beta\alpha,\ \alpha\gamma;\quad \gamma\beta,\ \alpha\beta;\quad \gamma\alpha,\ \beta\gamma;$

(4) $\alpha\beta,\ \gamma\alpha;\quad \gamma\beta,\ \beta\alpha;\quad \alpha\gamma,\ \beta\gamma;$

Case I :—

Let α, β, γ be actual points.

(*i*) Let the lines a, b, c be taken in directed pairs as group (1). The symmetric of $\beta\alpha$ and $\gamma\alpha$ is the internal bisector of the angle between b and c; so the symmetrics of $\gamma\beta, \alpha\beta$ and $\alpha\gamma, \beta\gamma$ are the internal bisectors of the angles between c, a and a, b. Hence the symmetrics are concurrent (*Theorem I*).

(*ii*) If now the lines be taken in directed pairs as in group (2), the symmetric of $\beta\alpha, \gamma\alpha$ is the internal bisector of the angle between b and c; but the symmetrics of $\beta\gamma, \alpha\beta$ and $\alpha\gamma, \gamma\beta$ are the external bisectors of the angles between c, a and a, b. So the three symmetrics are concurrent (*Theorem II*).

(*iii*) If the lines be taken in directed pairs as in group (3) or (4) we have a repetition of (*ii*).

Case II :—

Let α, β, γ be ideal points and suppose every two of the lines a, b, c lie on the same side of the third. In this case no straight line can meet all the three lines at actual points.

Let AA', BB' and CC' be the common perpendiculars between b, c; c, a and a, b and let P, Q, R be their middle points and p, q, r be the perpendiculars through P, Q, R to AA', BB' and CC' respectively.

(*i*) When the lines a, b, c are taken in directed pairs as in group (*i*), the symmetrics of $\beta\alpha$, $\gamma\alpha$; $\gamma\beta$, $\alpha\beta$ and $\alpha\gamma$, $\beta\gamma$ are p, q and r respectively.

Suppose q and r meet at an actual point O. Perpendiculars OM and ON on the sides b and c are equal being each equal to the perpendicular OL on a. Therefore the symmetric p passes through O. Thus the three symmetrics p, q, r are concurrent at an actual point O.

If however q and r meet at an ideal point, they have a common perpendicular O_2O_3 (fig. 6). Now O_2O_3 cannot meet b or c, for if it meets b, it must meet a, but it is evident that it cannot meet a since if it meets a it cannot meet q. Let $O'L$, $O''M$, $O'''N$ be the common perpendiculars between O_2O_3 and a, b, c. Then $O''M = O'L = O'''N$. Therefore the common perpendicular to O_2O_3 and AA' bisects AA', that is, PO_1 is perpendicular to O_2O_3. Thus p, q, r have a common perpendicular, that is, the three symmetrics have a common ideal point.

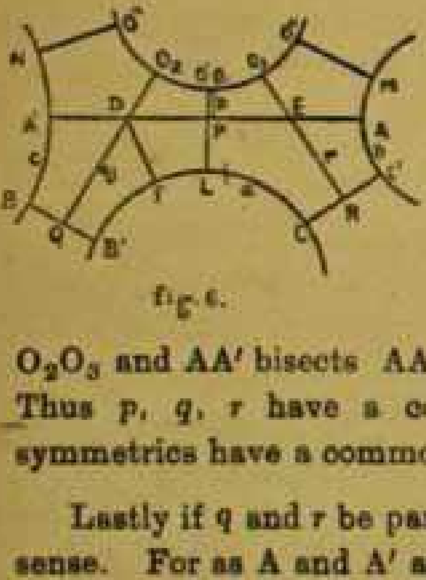


fig. 6.

Lastly if q and r be parallel p is also parallel to them in the same sense. For as A and A′ are points on opposite sides of q as well as of r, the line AA' must meet q and r at some points D and E (fig. 6). Let DF and EG be the perpendiculars from D and E on a. Then $DF < DG < DE + EG = DA$, $\therefore$ $DA' = DF < DA$. Similarly $EA = EG < EA'$. Hence P lies between D and E. Now p cannot meet q at an actual point as then r would pass through the same point and consequently could not be parallel to q; likewise p cannot meet r at an actual point. Thus r falls between the parallel lines q and r but does not meet either. Hence p must be parallel to q and r in the same sense. The three symmetrics are therefore concurrent at an improper point.

(*ii*) If now the lines be taken in directed pairs as in group (2), the symmetric of $\beta\alpha$, $\gamma\alpha$ is the line p; but the symmetrics of $\beta\gamma$, $\alpha\beta$

and $\alpha\gamma$, $\gamma\beta$ are the points Q and R. We are to show therefore that p is perpendicular to the join of Q and R.

The line QR (fig. 7) cannot meet any of the sides a, b, c, for if it meets one, it meets all the three which is impossible. If $O'L$, $O''M$, $O'''N$ be the common perpendiculars between QR and the sides a, b, c, it is easy to see that $O''M = O'L = O'''N$; therefore the common perpendicular between AA′ and QR bisects AA′, thus p is perpendicular to QR.

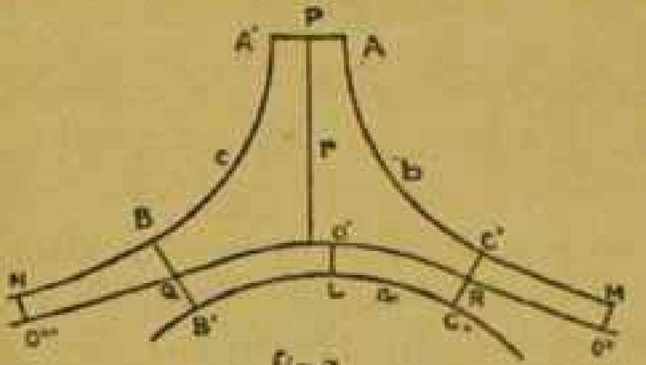


fig. 7.

(*iii*) If the lines be taken in directed pairs as in group (3) or (4) we get a repetition of (*ii*).

Case III: —

Let α, β, γ be ideal points and suppose the lines a, b, c be so related that one of them a has b and c on opposite sides.

Let AA′, BB′, CC′ be the common perpendiculars to the line pairs (b,c), (c,a) and (a,b); P, Q, R their middle points and p, q, r the perpendiculars through P, Q, R to AA′, BB′ and CC′ respectively (fig. 8).

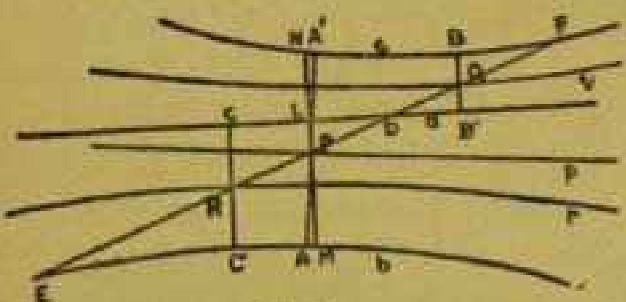


fig. 8

(*i*) If the lines be taken in directed pairs as in group (1), the symmetric of $\beta\alpha$ and $\gamma\alpha$ is the point P, but the symmetrics of $\gamma\beta$, $\alpha\beta$ and $\alpha\beta$, $\beta\gamma$ are the lines q and r. We are to show therefore that the common perpendicular to q and r passes through P.

Let the common perpendicular to q and r meet the sides a, b, c at L, M, N. LMN being perpendicular to q, $\measuredangle BNL = \measuredangle B'LN$. Similarly $\measuredangle C'ML = \measuredangle CLM$. But $\measuredangle B'LN = \measuredangle CLM$. $\therefore \measuredangle BNL = \measuredangle C'ML$. Therefore LMN passes through the middle point of AA′, that is, through the point P.

(*ii*) The lines being taken in directed pairs as in group (2), the three symmetrics are the points P, Q, R. We are to show therefore in this case that P, Q, R are collinear.

Let the line QR meet the three sides a, b, c in the points D, E, F. Then $\sphericalangle$BFQ$=\sphericalangle$B'DQ$=\sphericalangle$CDR$=\sphericalangle$C'ER. $\therefore$EDF bisects AA', that is, P lies on QR.

(*iii*) If the lines be taken in directed pairs as in group (3) or (4) we get a repetition of (*i*).

Case IV :—

Let α, β, γ be each an improper point.

Let p be the paralled to b and c which is equidistant from both, q the parallel to c and a equidistant from c and a and r the parallel to a and b equidistant from a and b (fig. 9).

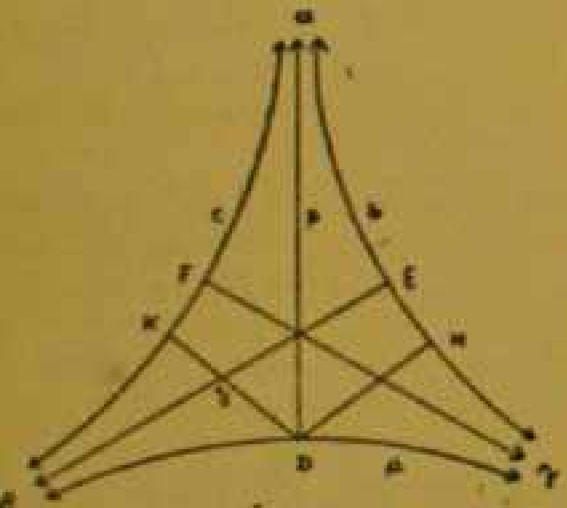


fig 9.

(*i*) Suppose the lines are taken in directed pairs as in group (1). The symmetrics of $\beta\alpha$, $\gamma\alpha$, $\gamma\beta$, $\alpha\beta$ and $\alpha\gamma$, $\beta\gamma$ are p, q, r respectively. We are to show that p, q, r are concurrent.

As q is parallel to c and a it cannot cut either again; it must therefore meet b at an actual point E and r must therefore cut q at an actual point O. The perpendiculars OM and ON on b and c are equal, each being equal to the perpendicular OL on a. Therefore O lies on p. Hence the three symmetrics pass through the same point O.

(*ii*) If the lines be taken in directed pairs as in group (2), the symmetric of $\beta\alpha$, $\gamma\alpha$ is the line p but the symmetrics of $\beta\gamma$, $\alpha\beta$ and $\alpha\gamma$, $\gamma\beta$ are the improper points β and γ so that their join is the line a. We are to show that p is at right angles to a.

The line p must meet a at an actual point D. The perpendiculars DH and DK on b and c are equal; hence $\sphericalangle a$DH and $\sphericalangle a$DK are equal being angles of parallelism for equal distances DH and DK. For a like reason $\sphericalangle p$DK$=\sphericalangle p$DH. Therefore p is at right angles to a.

(*iii*) The other arrangements of the directed pairs will give repetitions of (*ii*).

Case V :—

Let α be an actual point, β an ideal and γ an improper point.

Let A represent the actual point α, BB$'$ the common perpendicular between c and a represent the ideal point β and C represent the improper point γ, let p and p' be the internal and external bisector of the angle BAC, q the perpendicular to BB$'$ through its middle point Q and r the parallel to a and b which is equidistant from them (fig. 10).

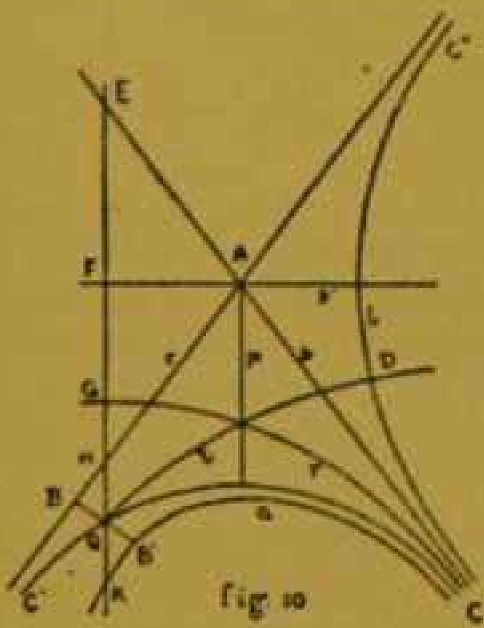


fig. 10

(*i*) If the arrangement of the directed pairs be as in group (1) the three symmetrics are p, q, r and the proof of their concurrency proceeds on the lines of the proof for this arrangement in the other cases.

(*ii*) Let the arrangement of the directed lines be as in group (2). The symmetric of $\beta\alpha$, $\gamma\alpha$ is p; but the symmetrics of $\beta\gamma$, $\alpha\beta$ and $\alpha\gamma$, $\gamma\beta$ are the points Q and C. We are to show that p is perpendicular to QC which is the parallel through Q to a and b.

If QC be produced through Q to C$'$ on the side of BB$'$ away from C, QC$'$ will be parallel to AB in the direction AB as Q is the middle point of BB$'$. Hence p is perpendicular to QC [*Case IV* (*ii*)].

(*iii*) If the lines be taken in directed pairs as in group (3), the symmetric of $\beta\alpha$, $\alpha\gamma$ is p', that of $\gamma\beta$, $\alpha\beta$ is q while that of $\gamma\alpha$, $\beta\gamma$ is the point C. We are to prove therefore that the common perpendicular to p' and q passes through the improper point C, that is, is parallel to a and b.

Through Q draw QC parallel to a and b and on q cut off the length QD corresponding the angle of parallelism DQC. The line l through D perpendicular to q will be parallel to a and b in the same sense and when produced through D will meet BA produced at an improper point C$''$. p' will therefore be perpendicular to l [*Case IV* (*ii*)]. Hence the common perpendicular to p' and q is parallel to a and b.

(*iv*) Lastly let the lines be taken in directed pairs as in group (4); the symmetric of $\alpha\beta$, $\gamma\alpha$ is the line p', that of $\gamma\beta$, $\beta\alpha$ is the point

Q, whereas of $\alpha\gamma$, $\beta\gamma$ is the line r. We are to show that the common perpendicular to p' and r passes through Q.

Let the perpendicular through Q to r meet the lines b, p', r, c and a at the points E, F, G, H and K respectively, then $\measuredangle QKB' = \measuredangle QHB$ and $\measuredangle QKB' = \measuredangle QEA$; $\therefore AEH = \measuredangle AHE$. Hence p is perpendicular to QE.

If two of the points α, β, γ be actual and the third ideal or improper, or two of them ideal and the third actual or improper, or finally, if two of them be improper and third actual or ideal, the proof of the theorem proceeds on similar lines and requires no further special investigation.

Moreover when one at least of the three points α, β, γ is an ideal one, the triangle formed by the three lines a, b, c may either be such that any two of the three lines a, b, c lie on the same side of the third, or that one of the lines a, b, c has the other two on opposite sides of it but as the proof of theorem in all these varied cases requires no new principles, the different cases are not separately discussed.

We shall now extend the theorem of the perpendiculars (*Theorem III*) to the case of a triangle of which the vertices may be actual, ideal or improper.

Theorem V :—*In a triangle formed by any three co-planar lines a, b, c intersecting not necessarily in actual points the perpendiculars p, q, r, from the vertices α, β, γ—actual, ideal or improper—on the opposite sides are concurrent ; the concurrency being interpreted as in Theorem IV.*

When α is an ideal point the perpendicular p on a is the common perpendicular between α and the line a, if a and α be non-intersecting; when a and α meet at an actual or improper point, the common perpendicular reduces to this actual or improper point. If α be an improper point, the perpendicular p is the perpendicular to a which is parallel to b and c in the same sense.

Case I :—

Let α, β, γ be actual points.

The theorem for this case has been already proved (*Theorem III*). The following is a more general proof and is applicable to almost all the cases.

Let A, B, C represent the three actual points α, β, γ and suppose the perpendiculars BE and CF on AC and AB meet at O (fig. 11). Draw the lines EL and FM making $\measuredangle$BEF = $\measuredangle$BEL and $\measuredangle$CFE = $\measuredangle$CFM. Then BE, AC and CF, AB are the internal and external bisectors of the angles FEL and EFM. It follows from the theorem of symmetrics (*Theorem IV*) that as AC and FC meet at an actual point C, EL and FM must also meet at an actual point D and OD is the internal bisector of the angle EDF. Again from the same theorem it follows that the line through D perpendicular to OD must pass through B and C; in other words D must lie on BC and OD is perpendicular to BC. But OD passes through A as A is the point of intersection of the external bisectors of the angles E and F. Hence the perpendicular from A on BC passes through O. The three perpendiculars are therefore concurrent. Similar proof holds if the point O be ideal or improper.

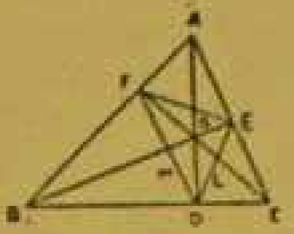


fig 11

Case II :—

Let α, β, γ be ideal points and suppose every two of the lines a, b, c be on the same side of the third (fig. 12).

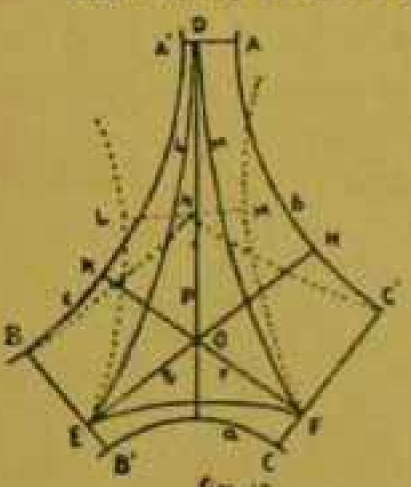


fig. 12.

Let BB′ and CC′ be the common perpendiculars between c, a and a, b representing the points β and γ and EH and FK the common perpendiculars q and r between the lines β, b and γ, c and let and r meet at the point O. Join EF and draw EL and FM making $\measuredangle$HEL = $\measuredangle$HEF and $\measuredangle$KFM = $\measuredangle$KFE. Suppose EL and FM meet at the actual point D. Then DO bisects $\measuredangle$EDF and it follows from *Theorem IV* that OD is the perpendicular to a. If now ADA′ be drawn at right angles to OD, from the same theorem it follows that ADA′ is perpendicular to both b and c, so that ADA′ is the common perpendicular between b and c. Hence OD is the common perpendicular between α and a. The three perpendiculars p, q, r are therefore concurrent.

If EL and FM do not meet at an actual point, they must be either non-intersecting or parallel. Suppose in the first place that

they are non-intersecting. Let LM represent the common perpendicular between them and N be the middle-point of LM. From *Theorem IV* it follows that the perpendicular between EH and CC′ must pass through N. Similarly c must pass through N. Thus b and c meet at an actual point N which is contrary to hypothesis. Hence EL and FM cannot be non-intersecting. If now EL and FM are parallel, it can be shown in like manner, that b and c must be parallel to them in the same sense which is against hypothesis. Hence EL and FM must meet at an actual point D.

Case III :—

Let α, β, γ be ideal points and the lines a, b, c be so related that two of them, b and c, lie on opposite sides of the third line a. Then the common perpendicular AA′ between b and c must meet a at some point P.

(*i*) Suppose the common perpendiculars BB′ and CC′ between c, a and a, b meet b and c respectively at the points Q and R (fig. 13). We are to show that P, Q and R are collinear.

Join A′C. Draw the line A′L on the side of AA′ away from C making ∢AA′L = ∢AA′C. Similarly draw the line CM on the side of CB′ remote from A′ making ∢B′CM = ∢B′CA′. Then AA′, A′B and B′C, CC′ are the internal and external bisectors of the angels LA′C and MCA′. If possible let A′L and CM meet at D. The line DP and the perpendicular EDF through D to DP are the internal and external bisectors of the angle ADC. From *Theorem IV* it follows that EDF is perpendicular to both the intersecting lines BB′ and C′A which is absurd. Hence A′L and CM cannot meet at an actual point. Nor can they be parallel, for from the same theorem it would follow that BB′ and C′A would be parallel each being parallel to the lines A′L and CM in the same sense and this is impossible. Hence A′L and CM are non-intersecting.

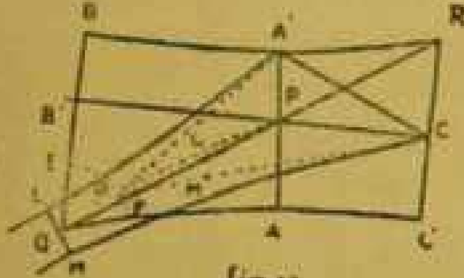


fig. 13.

From the same theorem it is further evident that Q is the middle point of the common perpendicular LM between A′L and CM. Hence the line through Q perpendicular to LM must pass through p, the point of intersection of the internal bisectors of the angles

CA′L and A′CM and also through R, the point of intersection of the external bisectors of the same angles. Thus P, Q, R are collinear.

(*ii*) Let BB′, *b* and CC′, *c* be non-intersecting lines and *l*, *m* be the common perpendiculars between them. We are to show that the common perpendicular to *l* and *m* must pass through P.

If we consider the ideal triangle of which the sides are *b*, *m* and BB′, the theorem follows from *Case II*.

(*iii*) Let now BB′, *b* and C′C, *c* be pairs of paralallel lines (fig. 14), and *n* be a line which is parallel to both BB′ and *c* and therefore to *b* and C′C in the same senses. We are to show that P lies on *n*.

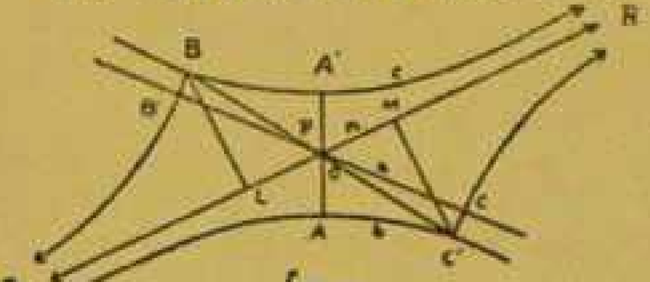


fig 14.

Let BL and C′M be the perpendiculars from B and C′ on the line *n*. It is evident BL = C′M being the lengths corresponding to angles of parallelism each equal to half of a right angle. Hence BC′ is bisected at the point O where it meets LM. Therefore the perpendicular through O to *b* will also be perpendicular to *c*; similarly the common perpendicular to BB′ and CC′ will pass through O and hence O coincides with P. Thus P lies on *n*.

Case IV :—

When α, β, γ are improper points, the three symmetrics *p*, *q*, *r* are the perpendiculars from α, β, γ on *a*, *b*, *c* respectively [*Theorem IV*, *Case IV*, (*ii*) (*iii*), (*iv*)]. Hence the three perpendiculars are concurrent [*Theorem IV*, *Case IV*, (*i*)].

GEOMETRICAL INVESTIGATIONS ON THE CORRESPONDENCES BETWEEN A RIGHT-ANGLED TRIANGLE, A THREE-RIGHT-ANGLED QUADRILATERAL AND A RECTANGULAR PENTAGON IN HYPERBOLIC GEOMETRY

BY

S. MUKHOPADHYAYA (1922) *

THEOREM 1.

A B C is a triangle right-angled at C on a given hyperbolic plane. AU is parallel to CB and DV is parallel to AB and perpendicular to AC meeting AC at D. From AU is cut off AE equal to AB and from DV is cut off DF equal to CB. Then EF is the common perpendicular to AU and DV. See fig. 1.

Produce CB to X and AB to Y. Join BE and BF. Let G, H, K and L be the mid-points of CD, EF, EB and BF, respectively. Suppose p is the common parallel to AY and CX. Let AK and GL, which bisect BE and BF, respectively, at right angles, meet p at K′ and L′, respectively, evidently at right angles.

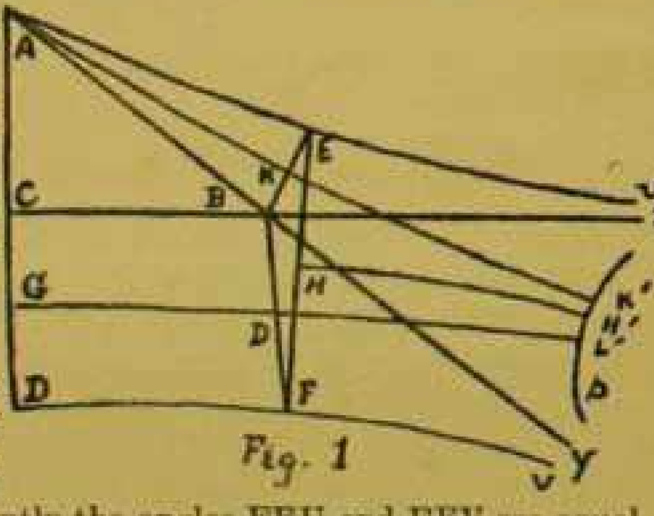


Fig. 1

It follows that the perpendicular bisector of EF also meets p at right angles at H′, and consequently the angles FEU and EFV are equal.

It may be observed that E, B, F cannot be collinear, for then the quadrilateral KK′L′L would be four-right-angled. Similarly B will fall between EF and AD, as otherwise the pentagon KK′L′LB would have its angle-sum greater than six right angles.

* From Bulletin, Calcutta Mathematical Society, Vol. 13, 1922.

Produce AD to M (see fig. 2) making DM equal to AC. Join MF and produce it to Z. Then the triangles ABC and MFD are congruent and MZ is parallel to CX, because AY is parallel to DV.

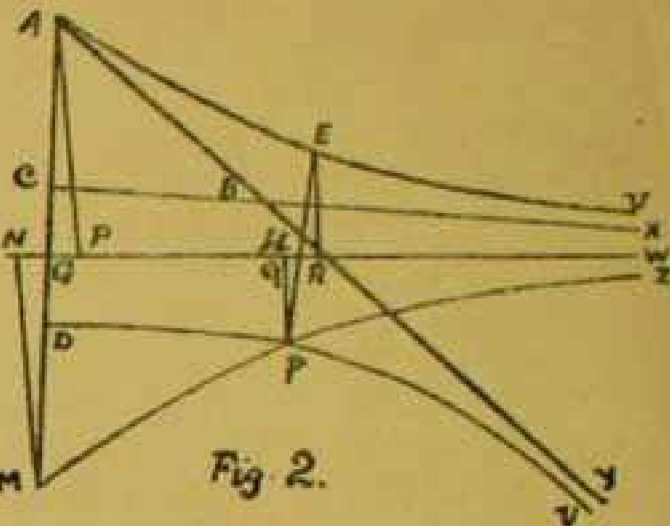


Fig. 2.

Draw GW parallel to CX and let AP, ER, MN, FQ be perendiculars to GW.

Then, because G is mid-point of AM, MN is equal to AP. Consequently the angle NMZ is equal to the angle PAU. Also MF is equal to AE. Therefore the quadrilaterals NMFQ and PAER are congruent. It follows that FQ is equal to ER, and GW passes through the mid-point H of EF.

Now as G is the mid-point of the common perpendicular CD to CX and DV, FD is parallel to WG and consequently the angle QFD is equal to the angle REU. Also the angles HFQ and HER are equal. Therefore the angles FEU and EFD are equal. But the angles FEU and EFD have been proved to be supplementary. Therefore each of them is a right angle.

Corollary 1.

If a and b denote the two sides of a right-angled triangle and c the hypotenuse and if λ, μ denote the angles opposite the sides a, b, then a three-right-angled quadrilateral exists of which the fourth angle is β and the sides reckoned in order from this angle are l, a, m', c.

This follows at once from fig. 2, if we observe that angle DAE is β, length of AD is l, length of DF is a, length of FE is m', being the distance of parallelism for angle EFZ which is complementary to angle DFM or ABC, and the length of AE is c. Thus ADFE is the three-right-angled quadrilateral whose existence has to be established.

The above simple synthetic proof of a fundamental theorem due to Lobatschewsky may prove of interest.

COROLLARY 2.

*Given a length l to construct the corresponding angle of parallelism λ. (Bolyai's classical construction.)**

Take a length AD equal to l. Draw DF at right angles to AD and of any length. Draw FE at right angles to DF and draw AE perpendicular from A on FE. Thus DF and AE are obtained. Construct a right-angled triangle with DF as base and AE as hypotenuse. The angle opposite to the base is the required angle λ, as is obvious from Theorem 1. See fig. 1.

THEOREM 2.

ABCD is a three-right-angled quadrilateral having right angles at A, B and C. A'U and C'V are parallel to AD and CD, respectively, and perpendicular to BC and BA, respectively, where A' lies on BC produced and C' lies on BA produced. From C'V is cut off C'E equal to AD and from A'U is cut off A'F equal to CD. See fig. 3.

Then EF is the common perpendicular to C'V and A'U.

Produce AD to X and CD to Y. Join ED and DF. Suppose p is the common parallel to DX and DY. Then the perpendicular bisectors of CA' and AC' which are also the perpendicular bisectors of DF and DE, respectively, evidently meet p perpendicularly. Consequently the perpendicular bisector of EF will also meet p perpendicularly. Therefore from symmetry the angles FEV and EFU are equal.

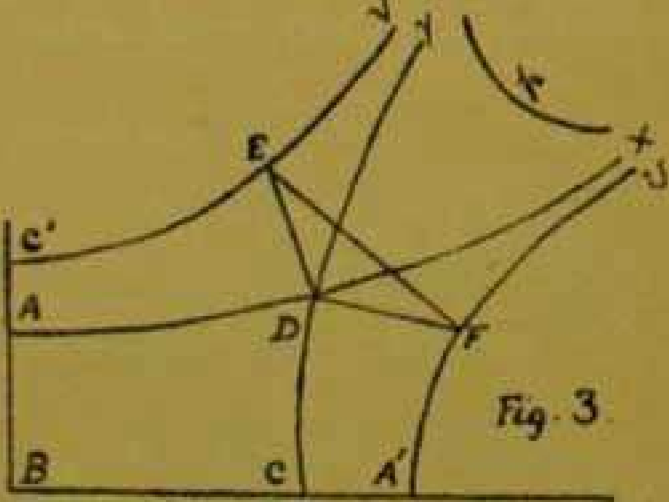


Fig. 3

It can be shewn, as in Theorem 1, that E, D, F cannot be collinear and that D lies on the side of EF on which B lies.

* For another proof see Liebmann, Nichteuclidishe Geometrie, 2nd edition, p. 35, or, Carslaw, Non-Euclidean Plane Geometry and Trigonometry, p. 73.

Produce BA′ to K, making A′K equal to BC, draw KL at right angles to BK and equal to BA. Join LF and produce it to Z. Then FZ is parallel to CY, because the quadrilaterals ABCD and LKA′F are congruent and AX is parallel to A′U. See fig. 4.

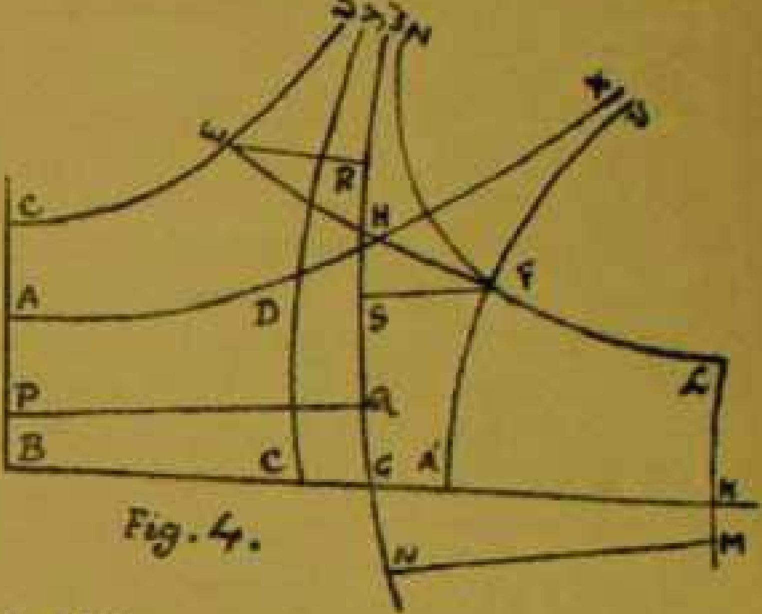


Fig. 4.

From G the mid-point of CA′ draw GW parallel to CY. Then GW will pass through the mid-point H of EF.

Draw MN and PQ common perpendiculars to the line-pairs LK, GW and AB, GW and ER, FS perpendiculars to GW.

Then obviously MN and PQ are equal and consequently the two three-right-angled quadrilaterals LMNW and C′PQW are congruent. Also because LF and C′E are each equal to AD, they are equal to each other. Hence it follows from the congruence of the pentagons NMLFS and QPC′ER that FS and ER are equal. Consequently GW bisects EF.

Then because G is the mid-point of A′C, and GW is parallel to CY, it follows that WG is parallel to UA′. Also because ER is equal to FS the angles FEC′ and EFA′ are equal. But these angles have been proved to be supplementary. Therefore each of them is a right angle.

Corollary 1.

If we write the five elements a, b, c, λ, μ of a right-angled triangle, in the order λ, μ, a, c, b there exists a rectangular pentagon whose sides in order are l, m, a', c, b'.

Suppose ABCD is the three-right-angled quadrilateral corresponding to a right-angled triangle with elements a, b, c, λ, μ, so that AB, BC, CD, DA are equal to m', a, l, c, and angle ADC is equal to β.

Construct the rectangular pentagon A'BC'EF as in Theorem 2. Then FA', A'B, BC', C'E are equal to l, m, a', c. The fifth side EF corresponds to angle of parallelism EFZ which is complementary to angle UFZ. But UFZ is equal to XDY, that is β, so that EF is equal to b'. See fig. 4.

Corollary 2.

With each vertex of the rectangular pentagon as origin we can reconstruct a three-right-angled quadrilateral and from this again a right-angled triangle. The sides of the rectangular pentagon may be written in order in five different ways:

$$l,\ m,\ a',\ c,\ b' \qquad (1)$$
$$m,\ a',\ c,\ b',\ l \qquad (2)$$
$$a',\ c,\ b',\ l,\ m \qquad (3)$$
$$c,\ b',\ l,\ m,\ a' \qquad (4)$$
$$b',\ l,\ m,\ a',\ c \qquad (5)$$

By identifying each of the sets (2), (3), (4), (5), with the set (1) we have five sets of possible values of a, b, c, λ, μ, including the given set, *viz.*:

$$a,\ b,\ c,\ \lambda,\ \mu$$

$$c',\ l',\ b',\ \mu,\ \frac{\pi}{2}-\alpha$$

$$b,\ m',\ l,\ \frac{\pi}{2}-\alpha,\ \gamma$$

$$l',\ a,\ m,\ \gamma,\ \frac{\pi}{2}-\beta$$

$$m',\ c',\ a',\ \frac{\pi}{2}-\beta,\ \lambda$$

We have thus the closed series of 5 associated right-angled triangles and the Engel-Napier analogies are shewn to possess a real geometrical basis in the rectangular pentagon.

The simple but highly important correspondences between a right-angled triangle and a rectangular pentagon, above pointed out, seem to have escaped the notice of previous investigators.

GENERAL THEOREM OF CO-INTIMACY OF SYMMETRICS OF A HYPERBOLIC TRIAD

BY

S. MUKHOPADHYAYA AND R. C. BOSE (1926)*

1. INTRODUCTION.

By a hyperbolic triad is meant a group of three elements (points or lines) lying upon a hyperbolic plane. The group may consist either of three lines, or two lines and a point, or two points and a line or three points, and the elements forming the group may be situated in any manner whatsoever on the plane. The scope of the presen' paper is to extend to all hyperbolic triads the well-known concurrency theorems of the angle bisectors and the right bisectors of the side of a triangle formed by three line elements meeting at three actual vertices.

The extension of the angle bisector theorem to all possible triads of linear elements, meeting at actual, improper or ideal vertices was effected by pure Geometry in a paper by S. Mukhopadhyay and Bhar published in the *Bulletin of the Calcutta Mathematical Society* [Vol. XII, No. 1, 1920-21]. They were first to introduce the concept of the symmetric between two directed lines, and to show that in certain cases it may be a point. The concurrency theorems of the angle bisectors of an ordinary triangle were then shown to be merely particular cases of the general theorem of concurrency of symmetrics between three directed lines.

In the present paper the concept of symmetric between a point and a line has been first introduced. By the introduction of this important concept which is claimed to be novel, the difficult problem of generalising the concurrency theorem of the right bisectors of the sides of a triangle so as to cover the cases when two or more of the sides do not meet at actual points has been completely solved. Again by the introduction of the concept of *intimacy* it has been possible to entirely abolish the ultra-geometrical concepts of improper, and ideal points, and at the same time to give to our theorems

* From Bulletin, Calcutta Mathematical Society, Vol. 17, 1926.

a larger scope for generality. Some other new terms and concepts have also been introduced which will be found in their proper places. The final result obtained is an elegant geometrical theorem of a highly general character which is applicable to all hyperbolic triads, and further combines in one the two distinct theorems of concurrency, already mentioned. The theorem as well as the numerous deductions which have been made from it, will, it is hoped, prove interesting to all lovers of Non-Euclidean Geometry.

2. LEMMA I.—*If P_1P_2N be perpendicular to O_1NO_2 and if P_1N and P_2N be complementary lengths, then a horocycle through P_1 and P_2 will touch O_1NO_2 at some point O_1 or O_2, such that O_1N or O_2N is complementary to the length $\frac{1}{2}(NP_1+NP_2)$.*

Let OX be a tangent to a horocycle at O and P_1N be perpendicular from any point P_1 on the horocycle on OX (Fig. i). Draw OY perpendicular to OX towards the same side on which P_1 lies, so that OY is an axis of the horocycle, drawn from O in the direction of parallelism. Draw P_1L an axis of the horocycle at P_1.

$$\text{Let } ON=x,\ P_1N=y,\ OP_1=2z,\ \angle LP_1O=\angle YOP_1=\phi$$

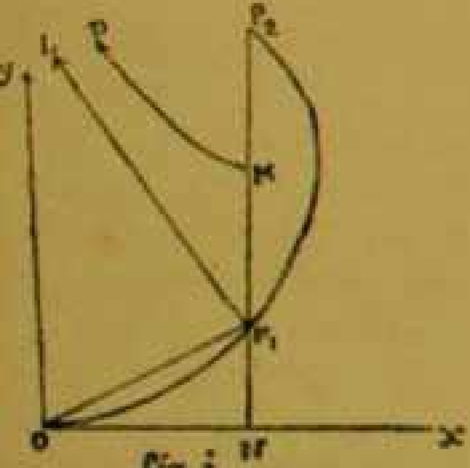


fig. i

Then evidently ϕ is the angle of parallelism for the distance z.

$$\therefore \quad \tan z=\cos\phi$$

$$=\sin P_1ON$$

$$=\frac{\sinh y}{\sinh 2z}$$

$$\text{or } \sinh y=2\sinh^2 z$$

$$\text{Again} \quad \cosh 2z=\cosh x\cosh y$$

$$\therefore \quad 1+\sinh y=\cosh x\cosh y \qquad \dots \quad (i)$$

$$\text{or } \cosh x=\operatorname{sech} y+\tanh y \qquad \dots \quad (ii)$$

Squaring (*i*) and simplifying, we get

$$\sinh^2 y-2\operatorname{cosech}^2 x\sinh y+1=0 \qquad \dots \quad (iii)$$

If y_1 and y_2 be two values of y corresponding to a given value of x, we have from (*iii*)

$$\sinh y_1 \sinh y_2=1.$$

Thus if NP_1 cuts the horocycle again at P_2, NP_1 and NP_2 are complementary.

From M the mid-point of P_1P_2 draw MD parallel to P_1L. Then MD is parallel to OY. It follows that the length ON is complementary to the length MN which is $\frac{1}{2}(NP_1+NP_2)$.

Corollary.—If N, P_1, M, P_2 be points taken in order upon a straight line such that M is the mid-point of P_1P_2 and lengths NP_1 and NP_2 are complementary, then $\cosh MP_1=\sinh MN$.

From equation (*i*), $\cosh x=\dfrac{1+\sinh y_1}{\cosh y_1}=\dfrac{1+\sinh y_2}{\cosh y_2}$;

$$\therefore \quad \sinh(y_1-y_2)=\cosh y_1-\cosh y_2,$$

or
$$\cosh \tfrac{1}{2}(y_1-y_2)=\sinh \tfrac{1}{2}(y_1+y_2).$$

But $\qquad \frac{1}{2}(y_1-y_2)=MP_1$ and $\frac{1}{2}(y_1+y_2)=MN$

Hence $\qquad \cosh MP_1=\sinh MN.$

3. Definitions.—*The principal line of a pair of elements consisting of a point and a line.* Let P be a point and AB a line not passing through P. Draw PL perpendicular to AB meeting AB at L. Take P′ on PL such that P′L is complementary to PL, P and P′ lying on the same side of L. Let M be the mid-point of PP′. Then the line perpendicular to PP′ at M is defined to be the *principal* line of P and AB.

It follows from the corollary to Lemma I that $\cosh MP=\sinh ML$.

The principal point of a pair of elements consisting of a point and a line.—Let P be a point and AB a line not passing through it. Draw PL perpendicular to AB meeting AB at L. Take L′ on PL such that PL′ is complementary to PL, L and L′ lying on the same side of P. Let S be the mid-point of LL′. Then the point S is defined as the *principal point* of P and AB.

It follows from the corollary to Lemma I that $\cosh SL=\sinh SP$.

The middle parallel of two parallel lines.—The locus of points equidistant from two given parallel lines is a line parallel to both. This line is defined to be the *middle parallel* of the two given lines.

4. Lemma II.—*If Q be any point on the principal line of P and AB then $\cosh PQ=\sinh QD$ where D is the foot of the perpendicular drawn from Q on AB.*

Let P be a point and AB any line not passing through P. Let PL be drawn perpendicular to AB, L lying on AB. Let MQ be the principal line of P and AB, M lying on PL. Let D be the foot of the perpendicular drawn from Q on AB. Join QP (Fig. *ii*).

Then $\sinh QD = \sinh ML \cosh MQ.$

$= \cosh PM \cosh MQ.$

$= \cosh PQ.$

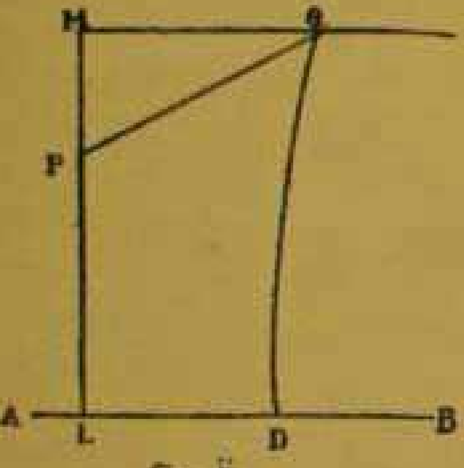


fig. ii

LEMMA III.—*If a line is perpendicular to the principal line of P and AB and if p is the length of the perpendicular from P on this line, then*

$\sinh p = \cos\phi$, 1, or $\cosh d$

according as the line intersects AB at an angle ϕ, is parallel to it, or possesses a common perpendicular of length d with it.

Let AB, P, L and M be as in Lemma II. Let QC be any line perpendicular to MQ the principal line of P and AB. Let PK be the perpendicular from P on QC such that $PK = p$. Let QC cut AB at an angle ϕ (Fig. *iii*), be parallel to it (Fig. *iv*), or possess a common perpendicular of length d with AB (Fig. *v*).

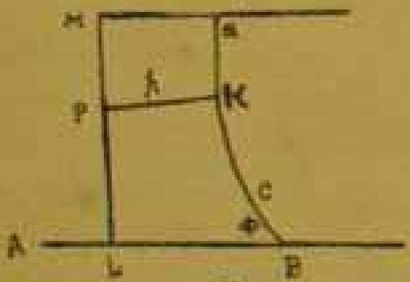


fig. iii

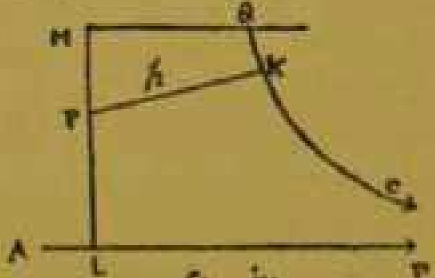


fig. iv.

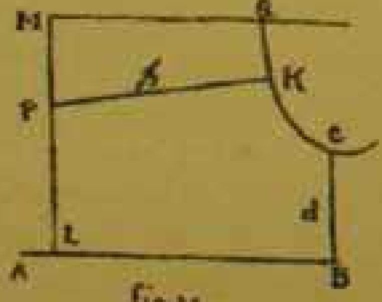


fig. v

Then $\cos\phi$, 1, or $\cosh d = \sinh ML \sinh MQ.$

$= \cosh PM \sinh MQ.$

$= \sinh p.$

LEMMA IV.—*If a line passes through the principal point of P and AB, and if p be the length of the perpendicular from P on it, then*

$$\sinh p = \cos \phi,\ 1,\ \text{or} \cosh d$$

according as the line intersects AB at an angle ϕ, *is parallel to it, or possesses a common perpendicular of length d with it.*

Let AB, P and L be as before. Let SH be any line through S the principal point of P and AB, itersecting AB at an angle ϕ (Fig. *vi*), parallel to it (Fig. *vii*), or possessing a common perpendicular of length d with it (Fig. *viii*). Let PR be the perpendicular from P on SH such that $PR = p$.

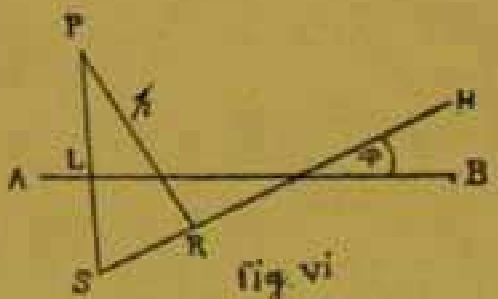


fig. vi

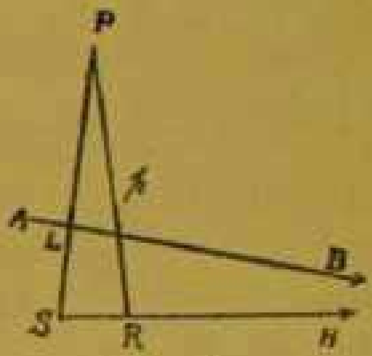


fig. vii

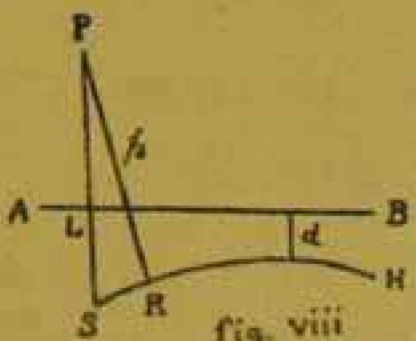


fig. viii

Then $$\cosh d,\ 1,\ \text{or} \cos \phi = \cosh SL \sin LSH,$$
$$= \sinh SP \sin LSH,$$
$$= \sinh p.$$

5. LEMMA V.—*If PM be a perpendicular to AB from a point P lying in AB and if p be the length of the perpendicular from P on any line parallel to PM (in the sense PM) then*

$$\sinh p = \cos \phi,\ 1,\ \text{or} \cosh d$$

according as the line intersects AB at an angle ϕ, *is parallel to it, or possesses a common perpendicular of length d with it.*

Let P be a point in the line AB. Let PM be drawn perpendicular to AB. Let CQ be a line parallel to PM (in the sense PM). Let CQ intersect AB at an angle ϕ (Fig. *ix*), be parallel to it (Fig. *x*), or possess a common perpendicular of length d with it (Fig. *xi*). Let PL be perpendicular to CQ such that PL $= p$.

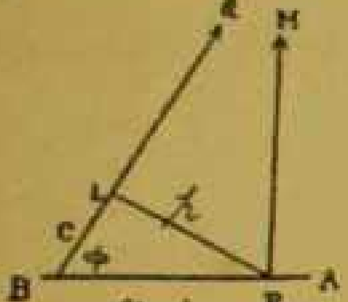


fig. ix.

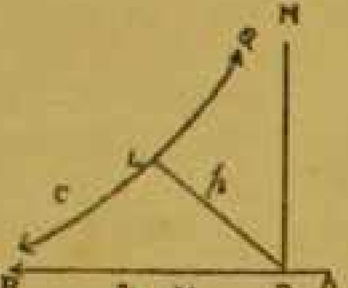


fig. X

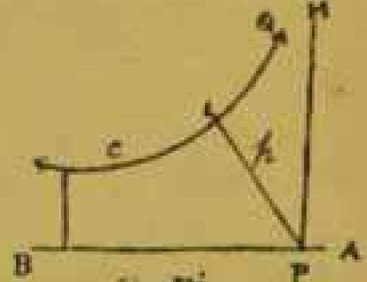


fig. xi

Then
$$
\begin{aligned}
\cos\phi,\ 1, \text{ or } \cosh d &= \cosh \mathrm{PL} \sin \mathrm{LPB}.\\
&= \cosh \mathrm{PL} \cos \mathrm{MPL}.\\
&= \cosh \mathrm{PL} \tanh \mathrm{PL}.\\
&= \sinh p.
\end{aligned}
$$

6. Intimacy and co-intimacy.

The elements we will deal with are the point, the line, and the horocycle, the last being representative of a conceptual point to which its axes converge. All horocycles with the same system of axes are equivalent.

A point and a line will be called *intimate* if the former lies on the latter. Two straight lines will be called *intimate* if one is perpendicular to the other. A straight line and a horocycle will be called *intimate* if the line is an axis of the horocycle. A horocycle may be regarded as *intimate* with itself and consequently with any equivalent horocycle.

The join of two elements is a third element intimate with both. Between any two distinct elements a unique join always exists. The join of two points is a straight line passing through both. The join of a point and a line is the straight line through the point, perpendicular to the given line (the point may lie on the line). The join of two intersecting straight lines is their point of intersection. The join of two parallel lines is any horocycle of which both the lines are axes. The join of two non-intersecting and non-parallel lines is the line perpendicular to both. The join of a point and a horocycle is that axis of the horocycle which passes through the point.

The join of two horocycles is the straight line which meets both horocycles at right angles and consequently is an axis to either horocycle. The join of a horocycle and a line, not its axis, is that axis of the horocycle which is perpendicular to the given line. The join of a horocycle and a line which is its axis may be taken to be the horocycle itself or any equivalent horocycle.

Any three elements will be called *co-intimate* if there is a common element which is intimate with each. Thus three straight lines passing through the same point are co-intimate as each of them is intimate with this point. Also three straight lines perpendicular to the same straight line are co-intimate. Again three straight lines parallel in the same sense are co-intimate as each of them is intimate with a common horocycle. Two lines and a point are co-intimate if a straight line through the point perpendicular to one of the lines is also perpendicular to the other line. Two points and a line are co-intimate if the straight line passing through the points is perpendicular to the line. Three points are co-intimate if they lie on the same straight line. Two lines and a horocycle are co-intimate if an axis of the horocycle is perpendicular to both the lines. Again two lines and a horocycle are co-intimate if both the lines are axes of the horocycle, for in this case each of the three given elements is intimate with the horocycle itself. A point, a line and a horocycle are co-intimate if the perpendicular from the point to the line, is an axis of the horocycle. Two points and a horocycle are co-intimate if the straight line through the points is an axis of the horocycle. Two horocycles and a line are co-intimate if the common axis of the two horocycles is perpendicular to the line. Two horocycles and a point are co-intimate if the common axis of the two horocycles passes through the point.

It would hardly be appropriate to call the three elements concurrent in all the above cases. We have therefore ventured to introduce the name co-intimacy to cover all these cases and hope that it will be acceptable to Mathematicians. S. Mukhopadhyaya has used already the expression '*range of intimacy (co-intimacy) of two curves*' for the set of points of intersection of the two curves (*Sir Asutosh Mookerjee Silver Jubilee* Vol. II, 1922).

7. Directed elements.

A point element or a line element may be taken in two opposite senses. With each point P we may associate a clockwise or a

counter-clockwise direction of rotation about the point. With each line AB we may associate either the direction AB or the direction BA.

We attach no sense to a horocyclic element. A point or a line taken with a particular direction associated with it we will call a *directed element*.

The sense of a directed line AB *relative* to a point P is defined to be clockwise or counter-clockwise according as the circuit PABP is clockwise or counter-clockwise.

Two directed points having the same sense are called *similarly directed*. They are called *oppositely directed* if they have opposite senses.

A directed point and a directed line are called *similarly directed* if the sense of the line relative to the point is the same as the sense of the point. If these senses are opposite, the point and the line are said to be *oppositely directed*.

Two directed lines parallel to one another are called *similarly directed* if the sense of each is the same as the sense of parallelism or opposite to it. They are said to be *oppositely directed* if the sense of one is the same as the sense of parallelism while the sense of the other is opposite to it.

Two directed lines with a common perpendicular are called *similarly directed* if they have the same sense relative to a point on this common perpendicular produced, while they are said to be *oppositely directed* if their senses relative to such a point are opposite.

8. The measure of divergence between two directed elements.

The *divergence* between two directed points at a distance d apart is measured by $-\cosh d$, or $+\cosh d$ according as the points are similarly or oppositely directed.

The *divergence* between a directed point and a directed line at a distance d from it is measured by $-\sinh d$, or $+\sinh d$ according as the point and the line are similarly or oppositely directed. If they are intimate the measure of divergence between them vanishes.

The *divergence* between two directed lines meeting at a point and making an angle δ with one another is measured by $\cos \delta$.

The *divergence* between two directed lines parallel to one another is measured by $+1$ or -1, according as they are similarly or oppositely directed.

The *divergence* between two directed lines with a common perpendicular of length d is measured by $+\cosh d$, or $-\cosh d$ according as the lines are similarly or oppositely directed.

If ρ be a directed element such that the measure of divergence between ρ and a given directed element α is the same as the measure of divergence between ρ and another directed element β then ρ is defined to be *equidivergent* with α and β. It is evident that if a directed element ρ is equidivergent with the directed elements α and β, as also with the directed elements α and γ, then ρ is equidivergent with β and γ.

9. Horocycles equidivergent with two directed elements.

A horocycle is said to be *equidivergent* with two directed points α and β if an equivalent horocycle passes through both α and β and if every directed point taken on this equivalent horocycle is either similarly directed to both α and β or is oppositely directed to both.

A horocycle is said to be *equidivergent* with a directed point α and a directed line β if an equivalent horocycle passing through α touches β and if every directed point taken on this equivalent horocycle is either similarly directed to both α and β or is oppositely directed to both.

A horocycle is said to be *equidivergent* with two directed lines α and β if an equivalent horocycle touches both and if every directed point taken on this equivalent horocycle is either similarly directed to both α and β or is oppositely directed to both. Again a horocycle is said to be *equidivergent* with two similarly directed parallel lines α and β, if both α and β are axes of the horocycle.

We shall now show that if H is a horocycle equidivergent with the directed elements α and β and also with the directed elements α and γ then H is equidivergent with β and γ.

In the first case suppose that α and β are not similarly directed parallel lines. Draw a horocycle H′ equivalent to H and passing through α if it is a point or touching α if it is a line. Since H is equidivergent with α and β, H′ passes through β if it is a point or touches β, if it is a line. Since H is equidivergent with α and γ, H′ passes through γ, if it is a point or touches γ, if it is a line. Again if P is any point on H′ similarly directed to α, it follows that P is similarly directed to β as well as to γ, and if Q is any point on H′ oppositely directed to α, it follows that Q is oppositely directed to β as well as to γ. Hence from definition H is equidivergent with β and γ.

Next suppose α and β to be similarly directed parallel lines. The horocycle touching both α and β is not in this case equidivergent with α and β, as any directed point on this horocycle is oppositely directed to β when it is similarly directed to α. In this case H is a horocycle which has both α and β as axes. Since H is equidivergent with α and γ, the latter must also be an axis of H and similarly directed to α. It follows that β and γ are both axes of H and are similarly directed. Hence from definition H is equidivergent with β and γ.

10. THE SYMMETRIC BETWEEN TWO DIRECTED ELEMENTS.

Between any two directed elements α and β there exists a unique element $\{\alpha\beta\}$ intimate with all elements (directed points, directed lines, or horocycles) equidivergent with α and β. $\{\alpha\beta\}$ is defined to be the *symmetric* between α and β.

(i) *The symmetric between two similarly directed points P and Q is the right bisector of PQ.*—Let p be the right bisector. Obviously any directed line or directed point intimate with p is equidistant from P and Q, and is either similarly directed to both P and Q or is oppositely directed to both. It is thus equidivergent with P and Q. Again if there be a horocycle H intimate with p (*i.e.*, having p as an axis) then a horocycle through P having the same system of axes as H, passes through Q. Since P and Q are similarly directed, every directed point on this second horocycle is either similarly directed with respect to both P and Q or is oppositely directed to both. H is then by definition equidivergent with P and Q.

(ii) *The symmetric between two oppositely directed points P and Q is the mid-point of PQ.*—Any directed line AB intimate with the mid-point is equidistant from P and Q. Since P and Q lie on opposite sides of AB the circuits PABP and QABQ have opposite senses. Hence if the sense of P is the same as the sense of the circuit PABP the sense of Q is the same as that of QABQ, while if the sense of P is opposite to the sense of PABP the sense of Q is opposite to that of QABQ. In every case therefore AB is similarly directed to both P and Q or oppositely directed to both. It follows that AB is equidivergent with P and Q.

(iii) *The symmetric between a directed point P and a line AB similarly directed to it is the principal line of P and AB.*—Let p be the principal line. Let Q be any directed point on p. Then Q must be on the same side of AB as P since p cannot intersect AB

as it possesses a common perpendicular with AB. Hence Q is either similarly directed to both P and AB or is oppositely directed to both. Lemma II then shows that Q is equidivergent with P and AB. Similarly it follows from Lemma III that any line intimate with p (*i.e.*, perpendicular to p) is equidivergent with P and AB. Again Lemma I shows that a horocycle through P having p as an axis touches AB. It is also obvious that every directed point on this horocycle is either similarly directed to both P and AB or oppositely directed to both, since all points on the horocycle lie on the same side of AB as P. It follows that any horocycle intimate with p is equidivergent with P and AB.

(*iv*) *The symmetric between a directed point P, and a line AB oppositely directed to it is the principal point of P and AB.*—Let S be the principal point. Taking into consideration the directions of the elements concerned, it follows at once from Lemma IV that any directed line intimate with S is equidivergent with P and AB.

(*v*) *The symmetric between a directed point P and a directed line AB intimate with it is a horocycle having as an axis the directed line PL, the sense of AB relative to L being the same as the sense of the directed point P.*—Let H be this horocycle. It follows from Lemma V that any directed line intimate with H is equidivergent with P and AB. Again a horocycle H′ equivalent to H and passing through P touches AB at P. All points on H′ lie on the same side of AB as L and therefore the senses of AB relative to every point on H′ is the same as the sense of P. It follows that every directed point on H′ is either similarly directed to both P and AB or oppositely directed to both. Hence all horocycles intimate with H and therefore equivalent to it, are equidivergent with P and AB.

(*vi*) *The symmetric between the directed lines OA and OB meeting at the point O is the external bisector of angle AOB.*

(*vii*) *The symmetric between two similarly directed parallel lines is a horocycle having both the lines as axes.*

(*viii*) *The symmetric between two oppositely directed parallel lines is their middle parallel.*

(*ix*) *The symmetric between two similarly directed lines with a common perpendicular is the mid-point of this perpendicular.*

(*x*) *The symmetric between two oppositely directed lines with a common perpendicular is the line besecting this perpendicular at right angles.*

Conversely it can be shown in every case that a directed point, a directed line or a horocyle equidivergent with the directed elements α and β is intimate with the symmetric $\{\alpha\beta\}$.

11. THEOREM.—*If α, β, γ be any three distinct directed elements (points or lines) on a hyperbolic plane, the symmetrics $\{\beta\gamma\}$, $\{\gamma\alpha\}$ and $\{\alpha\beta\}$ are co-intimate.*

First suppose $\{\gamma\alpha\}$ and $\{\alpha\beta\}$ have a point or a line element as their join. Call this element ρ and associate a particular direction with it. ρ is then a directed element intimate with the symmetric between the directed elements γ and α, ρ is therefore equidivergent with γ and α. Similarly ρ is equidivergent with α and β. It follows from Art. 8 that ρ is equidivergent with β and γ. Hence ρ must be intimate with the symmetric $\{\beta\gamma\}$. The symmetrics $\{\alpha\beta\}$, $\{\beta\gamma\}$, $\{\gamma\alpha\}$ are therefore co-intimate, each being intimate with the common element ρ.

Next suppose that the join of $\{\gamma\alpha\}$ and $\{\alpha\beta\}$ is a horocyclic element H. H is then equidivergent with γ and α being intimate with the symmetric between them. Similarly H is equidivergent with α and β. It follows from Art. 9 that H is equidivergent with β and γ. It must therefore be intimate with the symmetric $\{\beta\gamma\}$. Hence the symmetrics $\{\alpha\beta\}$, $\{\beta\gamma\}$, $\{\gamma\alpha\}$ are co-intimate, each being intimate with the common element H.

12. SUMMARY OF CASES.

The following are the more important cases of the general theorem proved.

Case I.—If a triad consists of three points A, B, C, then

(a) The right bisectors of the lines BC, CA and AB either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(b) The right bisector of BC meets at right angles the line joining the mid-points of CA and AB.

Case II.—If a triad consists of a straight line l and two points B and C lying on the same side of it, then

(a) The principal line of B and l, the principal line of C and l, and the right bisector of BC either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The principal line of B and l meets at right angles, the line joining the mid-point of BC with the principal point of C and l.

(*c*) The right bisector of BC meets at right angles, the line joining the principal point of B and l with principal point of C and l.

Case III.—If a triad consists of a straight line l *and two points B and C lying on opposite sides of it, then*

(*a*) The principal point of B and l, the principal point of C and l, and the mid-point of BC lie on the same straight line.

(*b*) The principal line of B and l, and the principal line of C and l possess a common perpendicular passing through the mid-point of BC.

(*c*) The right bisector of BC and the principal line of C and l possess a common perpendicular passing through the principal point of B and l.

Case IV.—If a triad consists of two points B and C and a line l *passing through one of the points (say B), and if BL is drawn perpendicular to* l, *lying on the same side of* l *as C, then*

(*a*) The right bisector of BC and the principal line of C and l are either both parallel to BL or possess a common perpendicular parallel to BL.

(*b*) The line joining the mid-point of BC with the principal point of C and l is parallel to BL.

(*c*) The perpendicular from the principal point of C and l to the right bisector of BC is parallel to LB.

(*d*) The perpendicular from the mid-point of BC to the principal line of C and l is parallel to LB.

Case V.—If a triad consists of a point A and two lines m *and* n *with a common perpendicular PQ (P lying on* m *and Q lying on* n*), and if A lies between* m *and* n, *then*

(*a*) The right bisector of PQ, the principal line of A and m and the principal line of A and n, either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The right bisector of PQ meets at right angles, the line joining the principal point of A and m with the principal point of A and n.

(*c*) The principal line of A and n meets at right angles, the line joining the mid-point of PQ with the principal point of A and m.

Case VI.—If a triad consists of a point A and two lines m and n with a common perpendicular PQ and if the line m lies between A and n, then

(*a*) The principal point of A and *m*, the principal point of A and *n* and the mid-point of PQ lie on the same straight line.

(*b*) The principal line of A and *m* and the principal line of A and *n* possess a common perpendicular passing through the mid-point of PQ.

(*c*) The right bisector of PQ and the principal line of A and *m* possess a common perpendicular passing through the principal point of A and *n*.

Case VII.—If a triad consists of two lines m and n with a common perpendicular PQ and a point A lying on m, and if AL be drawn perpendicular to m, L lying on the same side of m as n, then

(*a*) The right bisector of PQ and the principal line of A and *n* are either both parallel to AL or possess a common perpendicular parallel to AL.

(*b*) The line joining the mid-point of PQ with the principal point of A and *n* is parallel to AL.

(*c*) The perpendicular from the mid-point of PQ on the principal line of A and *n* is parallel to LA.

(*d*) The perpendicular from the principal point of A and *n* to the right bisector of PQ is perpendicular to LA.

Case VIII.—If a triad consists of two parallel lines m and n and a point A lying between them, then

(*a*) The principal line of A and *m*, the principal line of A and *n*, and the middle parallel of *m* and *n* are either parallel in the same sense, meet at a common point or are all perpendicular to a common line.

(*b*) The middle parallel of *m* and *n* meets at right angles the line joining the principal point of A and *m* with the principal point of A and *n*.

(*c*) The perpendicular from the principal point of A and *n* to the principal line of A and *n* is parallel to *m* and *n* in the same sense in which they are parallel to each other.

Case IX.—If a triad consists of two parallel lines m and n, and a point A such that m lies between A and n, then

(*a*) The principal line of A and m, and the principal line of A and n possess a common perpendicular, parallel to m and n in the same sense in which they are parallel to each other.

(*b*) The line joining the principal point of A and m with the principal point of A and n is parallel to m and n in the same sense in which they are parallel to each other.

(*c*) The principal line of A and m, and the middle parallel of m and n possess a common perpendicular passing through the principal point of A and n.

Case X.—If a triad consists of two parallel lines m and n and a point A lying on m, and if AL be drawn perpendicular to m, L lying on the same side of m as n, then

(*a*) The principal point of A and n lies on a line parallel to AL and to n (in the sense in which n is parallel to m).

(*b*) The principal line of A and n meets at right angles the line parallel to LA and to n (in the sense in which n is parallel to m).

(*c*) The middle parallel of m and n, and the principal line of A and n possess a common perpendicular parallel to AL.

(*d*) The perpendicular from the principal point of A and n to the middle parallel of m and n is parallel to LA.

Case XI.—If a triad consists of two lines OA and OB, meeting at the point O, and another point C lying in the angle AOB, then

(*a*) The internal bisector of $\angle$AOB, the principal line of C and OA and the principal line of C and OB, either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The internal bisector of $\angle$AOB meets at right angles the line joining the principal point of C and OA with the principal point of C and OB.

(*c*) The external bisector of $\angle$AOB, and the principal line of C and OA, possess a common perpendicular passing through the principal point of C and OB.

Case XII.—If a triad consists of two lines OA and OB meeting at a point O, and another point C lying on OA, and if CL be drawn perpendicular to OA, L lying on the same side of OA as B, then

(*a*) The internal bisector of $\angle$AOB, and the principal line of C and OB are either both parallel to CL or possess a common perpendicular parallel to CL.

(*b*) The perpendicular from the principal point of C and OB to the external bisector of the $\angle$AOB is parallel to CL.

(*c*) The external bisector of $\angle$AOB and the principal line of C and OB are either both parallel to LC or possess a common perpendicular parallel to LC.

(*d*) The perpendicular from the principal point of C and OB to the internal bisector of $\angle$AOB is parallel to LC.

Case XIII.—If a triad consists of three lines BC, CA and AB meeting at the points A, B, C, then

(*a*) The internal bisectors of the angles BAC, CBA and ACB meet at a point.

(*b*) The internal bisector of $\angle$BAC and the external bisectors of $\angle$CBA and $\angle$ACB, either meet at a point, are all parallel in the same sense, or are all perpendicular to the same straight line.

Case XIV.—If a triad consists of two parallel lines AL and BM, and a line AB meeting the two former lines at A and B, then

(*a*) The internal bisector of $\angle$LAB, the internal bisector of $\angle$MBA and the middle parallel of AL and BM meet at a point.

(*b*) The external bisector of $\angle$LAB, the external bisector of $\angle$MBA and the middle parallel of AL and BM either meet at a point, are all parallel in the same sense or are all perpendicular to the same straight line.

(*c*) The internal bisector of $\angle$LAB and the external bisector of $\angle$MBA possess a common perpendicular parallel to AL and AB.

Case XV.—If a triad consists of two lines AL and BM having a common perpendicular PQ, and a line AB meeting the two former lines at A and B, and if L and M lie on the same side of AB, then

(*a*) The internal bisectors of $\angle$LAB and $\angle$MBA, and the right bisector of PQ either meet at a point, are all parallel in the same sense or are all perpendicular to a common line.

(*b*) The internal bisector of $\angle$LAB and the external bisector of $\angle$MBA possess a common perpendicular passing through the mid-point of PQ.

Case XVI.—If a triad consists of two lines OP and OQ meeting at O and another line LM such that PL is a common perpendicular to OP and LM, and QM is a common perpendicular to OQ and LM, then

(*a*) The internal bisector of $\angle$POQ, the right bisector of PL, and the right bisector of QM meet at a point.

(*b*) The internal bisector of $\angle$POQ meets at right angles the line joining the mid-points of PL and QM.

(*c*) The external bisector of $\angle$POQ and the right bisector of PL possess a common perpendicular passing through the mid-point of QM.

Case XVII.—If a triad consists of three lines AB, CD and EF such that AC is a common perpendicular to AB and CD, DE is a common perpendicular to CD and EF, and FB is a common perpendicular to AB and EF, and if every two of the lines lie on the same side of the third, then

(*a*) The right bisectors of AC, DE and BF either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The right bisector of AC meets at right angles the line joining mid-points of BF and DE.

Case XVIII.—If a triad consists of three lines, a, b, c, every two of the lines possessing a common perpendicular, and two of the lines, say b and c, lie on opposite sides of a, then

(*a*) The mid-points of the three common perpendiculars lie on the same straight line.

(*b*) The right bisectors of any two of the common perpendiculars themselves possess a common perpendicular, which passes through the mid-point of the third common perpendicular.

TRIADIC EQUATIONS IN HYPERBOLIC GEOMETRY

BY

S. MUKHOPADHYAYA AND R. C. BOSE (1927) *

1. INTRODUCTION.

The present paper is an application and development of the principles explained and developed in the paper "*General Theorem of co-intimacy of Symmetrics*" published in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXVI, No. I, 1926 and should be read for a proper understanding along with that paper.† A short resume however is given of the principles explained and the notations used in that paper so that in a manner it can be followed independently of that paper. The '*triadic co-ordinates*' introduced and so named in this paper differ from '*Wierstrassian co-ordinates*' mainly in the fact that any of the elements whose co-ordinates are united in any equation may be indifferently a point or a line. Thus points and lines stand in a relation of unity and not of duality.

2. DEFINITIONS.

We will denote the *point*, the *line* and the *horocycle* as *basic elements*.

All horocycles having the same system of axes will be considered equivalent as representing the same basic element, which is a conceptual point at infinity to which all the axes converge.

A point or a line as a basic element can have *associated* with it two *directed elements* of opposite senses. The horocycle as a basic element stands quite *isolated* in this respect.

* From Bulletin, Calcutta Mathematical Society, Vol. 18, 1927.

† The use of oriented points and lines which was first made in the paper above referred to and has been maintained in this, forms a special feature of this paper. T. Takasu of the Tohoku Imperial University in an elegant paper "Natural Non-Euclidean Geometry, Doubly Oriented Points, Lines and Planes as Elements" published in the Tohoku Mathematical Journal of April, 1925, has developed the theory of orientation of points, lines and planes in Non-Euclidean 3-space.

To a basic line element can be associated two *directed line elements* having the same position but opposite senses, *i.e.*, directions of imaginary translation along them.

To a basic point element can be associated two *directed point elements* having the same position but opposite senses, *i.e.*, directions of imaginary rotation about them.

The two directed elements associated with a basic point or a basic line will be called its *orients*. Of these if one be called the *positive orient* the other will be called the *negative orient*.

If α be a basic element, a point or a line, its two orients will be denoted by α_+ and α_- and either of them by α_0.

Two basic lines will be called *intimate* if they are at right angles. A basic point and a basic line will be called *intimate* if the latter passes through the former. A basic line is *intimate* with a horocycle if the former is an axis of the latter. A horocycle will be called *intimate* with itself or any eqivalent horocycle. It may be observed that two basic points cannot be intimate, neither can a basic point and a horocycle be intimate under any circumstances.

The *join* * of two basic elements is a third basic element intimate with both. It will be observed that a unique join exists in every case. If α and β be any two basic elements then $(\alpha\ \beta)$ will represent their join. Similarly the join of γ with $(\alpha\ \beta)$ will be represented by $\{(\alpha\beta)\gamma\}$ and the join of $(\alpha\beta)$ with $(\gamma\delta)$ by $\{(\alpha\beta)'\gamma\delta)\}$ and so on.

Any three elements will be called *co-intimate* if there is a common element intimate with each.

The sense of a directed line relative to a point not lying on it may be clockwise or counter-clockwise. Similarly the sense of a directed point about the point itself may be clockwise or counter-clockwise.

If the senses of two directed points are both clockwise or both counter-clockwise they are said to be *similarly oriented*, but if one of the senses be clockwise and the other counter-clockwise they are said to be *oppositely oriented*.

If the senses of a directed line and a directed point be such that the sense of the former relative to the base of the latter and the sense of the latter itself are both clockwise or both counter-clockwise they are said to be *similarly oriented* but if these senses be opposite they are said to be *oppositely oriented*.

* For a summary of the various cases that arise see the paper referred to in the introduction

If two directed lines be parallel and the senses of both are in the direction of parallelism or opposite to it, they are said to be *similarly oriented*, but if one of the senses be in the direction of parallelism and the other against it they are said to be *oppositely oriented*.

Two directed lines with a common perpendicular are called *similarly oriented* if they have the same sense relative to a point on this common perpendicular produced, while they are said to be *oppositely oriented* if their senses relative to such a point are opposite.

A directed element is said to be intimate with a basic element when the base of the former is intimate with the latter.

Two directed elements are said to be intimate when their bases are intimate.

3. Divergence.

The *divergence* between two directed points at a distance d apart is measured by $-\cosh d$ or $+\cosh d$ according as the points are similarly or oppositely oriented.

The *divergence* between a directed point and a directed line at a distance d from it is measured by $-\sinh d$ or $+\sinh d$ according as the point and the line are similarly or oppositely oriented.

The *divergence* between two directed lines meeting at a point and making an angle δ with one another is measured by $\cos \delta$.

The *divergence* between two directed lines parallel to one another is measured by $+1$ or -1 according as they are similarly or oppositely oriented.

The *divergence* between two directed lines with a common perpendicular of length d is measured by $+\cosh d$ or $-\cosh d$ according as the lines are similarly or oppositely oriented.

If we denote *divergence* by *div*, then evidently we have

$$\text{div}\,(\alpha_+, \beta_+) = \text{div}\,(\alpha_-, \beta_-) = -\text{div}\,(\alpha_+, \beta_-) = -\text{div}\,(\alpha_-, \beta_+)$$

It should be noted that *the necessary and sufficient condition that two directed elements α_0 and β_0 are intimate is div $(\alpha_0 \beta_0) = 0$.*

4. Co-ordinates of elements referred to a self-intimate triad.

A triad of directed elements such that each is intimate with the other two, will be called a *self-intimate triad*.

Let ξ_0 and η_0 be two directed lines intimate with one another. Let ζ_0 be a directed point intimate with both ξ_0 and η_0. Then ξ_0,

η_0, ζ_0 from a self-intimate triad. If a_0 be any other directed element then

$$\text{div}\,(a_0\,\xi_0),\ \text{div}\,(a_0\,\eta_0),\ \text{div}\,(a_0\,\zeta_0)$$

will be called the *triadic co-ordinates* of a_0.

5. THE IDENTICAL RELATION SATISFIED BY THE CO-ORDINATES OF A DIRECTED ELEMENT.

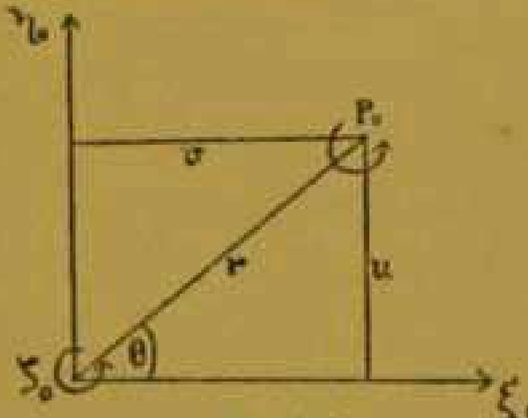


Fig.(1)

Case I.—Let P_0 be a directed point with co-ordinates x_1, y_1, z_1. Let r be the length of the radius vector drawn from ζ_0 to P_0 and θ the angle which this radius vector makes with ξ_0. Also let u and v be the lengths of the perpendiculars drawn from P_0 to ξ_0 and η_0 respectively. [See Fig. (1).]

Then

$$x_1=\text{div}\,(P_0\,\xi_0)=-\sinh u=-\sinh r\sin\theta \quad\ldots\quad (1)$$
$$y_1=\text{div}\,(P_0\,\eta_0)=\sinh v=\sinh r\cos\theta \quad\ldots\quad (2)$$
$$z_1=\text{div}\,(P_0\,\zeta_0)=-\cosh r \quad\ldots\quad (3)$$

Hence $$x_1^2+y_1^2-z_1^2=-1.$$

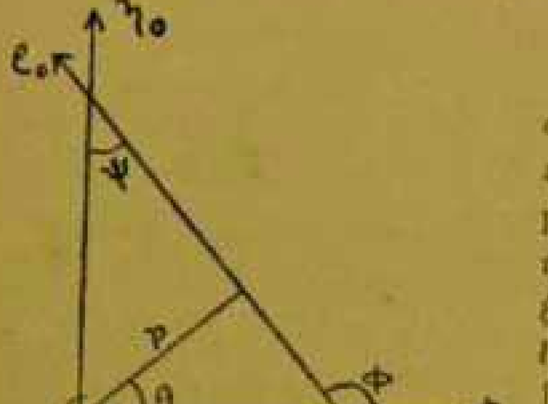


Fig.(2)

Case II.—Again let l_0 be a directed line with co-ordinates x_2, y_2, z_2. Let p be the length of the perpendicular from ζ_0 on l_0 and θ the angle this perpendicular makes with ξ_0. Let ϕ and ψ be the angles which l_0 makes with ξ_0 and η_0 respectively [See Fig. (2).]

Now
$$x_2=\text{div}\,(l_0\,\xi_0)=\cos\phi=-\sin\theta\cosh p \quad\ldots\quad (4)$$
$$y_2=\text{div}\,(l_0\,\eta_0)=\cos\psi=\cos\theta\cosh p \quad\ldots\quad (5)$$
$$z_2=\text{div}\,(l_0\,\zeta_0)=-\sinh p \quad\ldots\quad (6)$$

Hence $$x_2^2+y_2^2-z_2^2=+1.$$

If x, y, z be the co-ordinates of a directed element

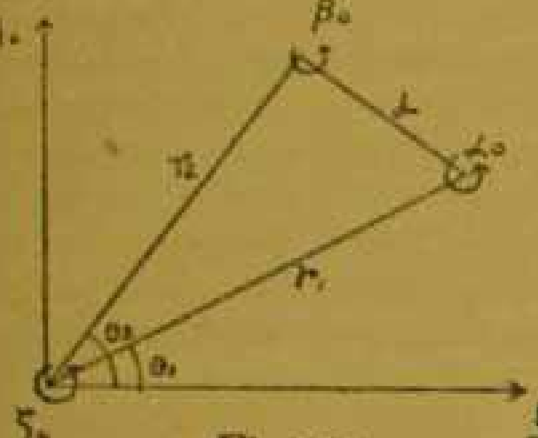
$$x^2+y^2-z^2=\mp 1 \quad\ldots\quad (7)$$

the upper or the lower sign being taken according as the element is a point or a line.

6. FUNDAMENTAL THEOREM.

If x_1, y_1, z_1 be the co-ordinates of a directed element α_0 and x_2, y_2, z_2 the co-ordinates of a directed element β_0, then

$$\operatorname{div}(\alpha_0\,\beta_0)=x_1x_2+y_1y_2-z_1z_2 \qquad \dots \quad (8)$$



Fig(3)

Case I.—Let α_0, β_0 be similarly directed points. Let r_1, r_2 be the lengths of the radius vectors from ζ_0 to α_0 and let θ_1, θ_2 be the angles which these radius vectors makes with ξ_0. Also let d be the distance between α_0 and β_0. [See Fig. (3).] Then

$x_1=-\sinh r_1 \sin\theta_1$,	$x_2=-\sinh r_2\sin\theta_2$	from	(1)
$y_1=\sinh r_1\cos\theta_1$,	$y_2=\sinh r_2\cos\theta_2$	from	(2)
$z_1=-\cosh r_1$,	$z_2=-\cosh r_2$	from	(3)

Therefore
$$\begin{aligned} x_1x_2+y_1y_2-z_1z_2 &= \sinh r_1 \sinh r_2 \cos(\theta_1-\theta_2) \\ &\qquad -\cosh r_1 \cosh r_2 \\ &=-\cosh d \\ &=\operatorname{div}(\alpha_0\,\beta_0). \end{aligned}$$

The same result would be seen to hold when α_0, β_0 are oppositely directed.

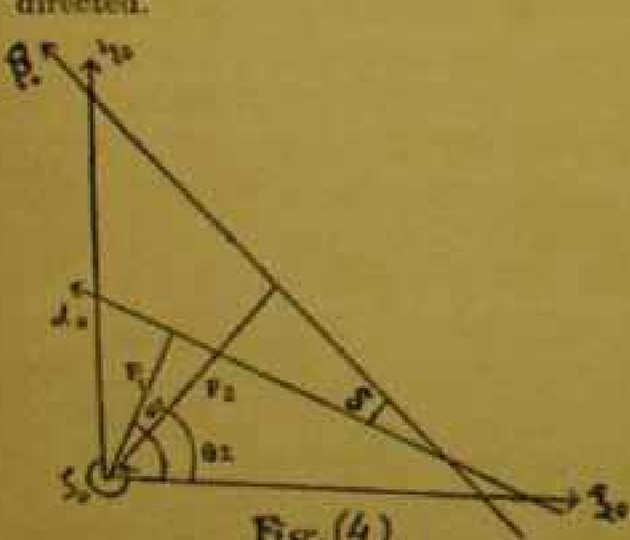


Fig.(4)

Case II.—Let α_0, β_0 be directed lines. Let p_1, p_2 be the lengths of the perpendiculars drawn from ζ_0 to α_0 and β_0 respectively and let θ_1, θ_2 be the angles which these perpendiculars make with ξ_0. Also let δ be the angle between α_0 and β_0. [See Fig. (4).] Then

$$x_1=-\sin\theta_1\cosh p_1, \quad x_2=-\sin\theta_2\cosh p_2 \text{ from} \qquad (4)$$

$y_1=\cos\theta_1\cosh p_1,\quad y_2=\cos\theta_2\cosh p_2$ from (5)

$z_1=-\sinh p_1,\quad z_2=-\sinh p_2$ from (6)

Therefore $x_1x_2+y_1y_2-z_1z_2=\cosh p_1\cosh p_2\cos(\theta_1-\theta_2)$

$$-\sinh p_1\sinh p_2$$

$$=\cos\delta=\operatorname{div}(\alpha_0\beta_0)$$

The same result would be seen to hold when α_0, β_0 are parallel or non-intersecting.

Case III.—Let α_0 be a directed point and β_0 a directed line. This case can be treated on lines similar to those adopted before.

7. EQUATION OF A BASIC ELEMENT.

We shall show that the co-ordinates of all directed elements intimate with a given basic element (point, line or horocycle) satisfy a linear equation. This equation will be called the *triadic equation* of the given element.

Case I.—Let the given basic element α be a point or a line.

Let α_0 be an orient of α. Let a, b, c be the co-ordinates of α_0. Let γ_0 be any directed element intimate with α, and let x, y, z be the co-ordinates of γ_0. Hence by definition γ_0 and α_0 are intimate. Thus div $(\alpha_0\gamma_0)$ vanishes. We then get from (8)

$$ax+by-cz=0 \qquad \ldots \quad (9)$$

The linear equation (9) is satisfied by the co-ordinates of all directed elements intimate with α.

Corollary (1).—If $ax+by-cz=0$ be the equation of α, a basic point or a basic line, and p, q, r are the co-ordinates of α_0, an orient of α, then

$$\frac{p}{a}=\frac{q}{b}=\frac{r}{c} \qquad \ldots \quad (10)$$

Corollary (2).—If $ax+by-cz=0$ be the equation of a point, we have

$$a^2+b^2<c^2 \qquad \ldots \quad (11)$$

but if the same is the equation of a line

$$a^2+b^2>c^2 \qquad \ldots \quad (12)$$

This result follows from (7) and (10).

Case II.—Let the given element α be a horocycle.

Let p_1, q_1, r_1 and p_2, q_2, r_2 be the co-ordinates of two fixed similarly directed parallel lines β_0 and γ_0 intimate with the horocycle. Let x, y, z be the co-ordinates of an arbitrary directed line δ_0 intimate with α. Then δ_0 is parallel to both β_0 and γ_0, and is either similarly directed to both β_0 and γ_0 or is oppositely directed to both. In the former case div $(\delta_0\beta_0)=$ div $(\delta_0\gamma_0)=+1$, while in the latter case div $(\delta_0\beta_0)=$ div $(\delta_0\gamma_0)=-1$. Hence from (8)

$$p_1x+q_1y-r_1z=p_2x+q_2y-r_2z$$

$$\text{or } (p_1-p_2)x+(q_1-q_2)y-(r_1-r_2)z=0 \qquad \dots \quad (13)$$

The linear equation (13) is then satisfied by all directed elements intimate with α.

Corollary.—If $ax+by-cz=0$ be the equation of a horocyclic element we have

$$a^2+b^2=c^2 \qquad \dots \quad (14)$$

For, $(p_1-p_2)^2+(q_1-q_2)^2-(r_1-r_2)^2$

$$=(p_1^2+q_1^2-r_1^2)+(p_2^2+q_2^2-r_2^2)$$
$$-2(p_1p_2+q_1q_2-r_1r_2)$$
$$=1+1-2$$
$$=0$$

8. The condition of intimacy of two elements whose equations are given.

Theorem.—If $a_1x+b_1y-c_1z=0$ and $a_2x+b_2y-c_2z=0$ be the equations of two basic elements α and β, the necessary and sufficient condition that α and β are intimate is

$$a_1a_2+b_1b_2-c_1c_2=0 \qquad (15)$$

Case I.—Let neither of α and β be horocyclic.

Let α_0 be an orient of α and β_0 an orient of β. Let p_1, q_1, r_1 be the co-ordinates of α_0 and p_2, q_2, r_2 the co-ordinates of β_0. Then from (10)

$$\frac{a_1}{p_1}=\frac{b_1}{q_1}=\frac{c_1}{r_1}=k_1 \text{ (say)}$$

and $$\frac{a_2}{p_2}=\frac{b_2}{q_2}=\frac{c_2}{r_2}=k_2 \text{ (say)}$$

Therefore $$a_1a_2+b_1b_2-c_1c_2=k_1k_2\ (p_1p_2+q_1q_2-r_1r_2)$$
$$=k_1k_2 \text{ div } (\alpha_0\beta_0)$$

This shows that the necessary and sufficient condition for the intimacy of α_0 and β_0, and hence of α and β is

$$a_1a_2+b_1b_2-c_1c_2=0$$

Case II.—Let α be horocyclic.

If β is intimate with α, then β must either be an equivalent horocycle in which case

$$a_1a_1+b_1b_2-c_1c_2=a_1^2+b_1^2-c_1^2=0 \text{ from (14)}$$

or β must be a line intimate with α. Let p, q, r be the co-ordinates of β_0, an orient of the line β. Then p, q, r satisfy the equation of α, so that

$$pa_1+qb_1-rc_1=0$$

but from (10), $$p/a_2=q/b_2=r/c_2,$$

so $$a_1a_2+b_1b_2-c_1c_2=0$$

Again if it is given that

$$a_1a_2+b_1b_2-c_1c_2=0 \qquad \dots \quad (i)$$

then if β is horocyclic we have in addition to (i)

$$a_1^2+b_1^2-c_1^2=0 \text{ from (14)}$$

$$a_2^2+b_2^2-c_2^2=0 \text{ from (14)}$$

Therefore $$\frac{a_1}{(b_1c_2-b_2c_1)}=\frac{b_1}{(c_1a_2-c_2a_1)}=\frac{c_1}{(a_1b_2-a_2b_1)}$$

and $$\frac{a_2}{(b_1c_2-b_2c_1)}=\frac{b_2}{(c_1a_2-c_2a_1)}=\frac{c_2}{(a_1b_2-a_2b_1)}$$

Thus

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

which shows that α and β are equivalent horocycles and therefore intimate.

Otherwise if β is a line then let p, q, r be the co-ordinates of β_0, an orient of β. Thus

$$p/a_2=q/b_2=r/c_2 \text{ from (10)}$$

Hence from (15)

$$pa_1+qb_1-rc_1=0$$

which shows that p, q, r satisfy the equation of the horocycle. Thus β_0 and hence β is intimate with α.

Corollary (1).—*If* $a_1x+b_1y-c_1z=0$, $a_2x+b_2y-c_2z=0$ *be the equations of two basic elements, the equation of their join is*

$$(b_1c_2-b_2c_1)x+(c_1a_2-c_2a_1)y-(a_1b_2-a_2b_1)z=0 \qquad \dots \quad (16)$$

The result follows at once from the fact that the join is intimate with both the given elements.

Corollary (2).—*The necessary and sufficient condition that the elements* α, β, γ *whose equations are*

$$a_1x+b_1y-c_1z=0$$

$$a_2x+b_2y-c_2z=0$$

$$a_3x+b_3y-c_3z=0$$

be co-intimate is the vanishing of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \dots \quad (17)$$

9. THE EQUATION OF THE SYMMETRIC BETWEEN TWO DIRECTED ELEMENTS.

If α_0 and β_0 be two directed elements, then there exists a unique basic element λ such that, all directed elements intimate with it are equidivergent with α_0 and β_0. λ is then defined to be the *symmetric* between α_0 and β_0.

Let p_1, q_1, r_1; p_2, q_2, r_2 be the co-ordinates of α_0 and β_0 respectively; then the equation of λ, the symmetric between them, is

$$(p_1-p_2)x+(q_1-q_2)y-(r_1-r_2)z=0 \qquad \dots \quad (18)$$

For if x_1, y_1, z_1 be the co-ordinates of any directed element γ_0 intimate with λ, then from (18)

$$(p_1-p_2)x_1+(q_1-q_2)y_1-(r_1-r_2)z_1=0$$

$$\text{or} \quad p_1x_1+q_1y_1-r_1z_1=p_2x_2+q_2y_2-r_2z_2$$

$$\text{or} \qquad \text{div}\,(a_0\gamma_0)=\text{div}\,(\beta_0\gamma_0)$$

To show that the symmetric is unique, we note that if there is any other element with equation

$$lx+my-nz=0 \qquad \dots \quad (ii)$$

which satisfies the definition of the symmetric, then

$$(p_1-p_2)x+(q_1-q_2)\,y-(r_1-r_2)z=0 \qquad \dots \quad (iii)$$

is satisfied for all values of x, y, z which satisfy (*ii*). Whence the equations (*ii*) and (*iii*) must be identical.

It has been shown in the paper referred to in the introduction that the symmetric between

(*i*) *Two similarly directed points P and Q, is the right bisector of PQ.*

(*ii*) *Two oppositely directed points P and Q, is the mid-point of PQ.*

(*iii*) *A directed point P and a line AB similarly directed to it, is the principal line * of P and AB.*

(*iv*) *A directed point P and a line AB oppositely directed to it, is the principal point of P † and AB.*

(*v*) *Two similarly directed parallel lines, is a horocycle having both lines as axes.*

* The principal line of P and AB is defined as follows :—Draw PL perpendicular to AB meeting AB at L. Take P′ on PL such that P′L is complementary to PL, P and P′ lying on the same side of L. Let M be the mid-point of PP′. Then the line perpendicular to PP′ at M is defined to be the principal line of P and AB.

† The principal point of P and AB is defined as follows :—Draw PL perpendicular to AB meeting AB at L. Take L′ on PL such that PL′ is complementary to PL, L and L′ lying on the same side of P. Let S be the mid-point of LL′. Then S is defined to be the principal point of P and AB.

(vi) *Two oppositely directed parallel lines, is their middle parallel.**

(vii) *The directed lines OA and OB meeting at O, is the external bisector of the angle AOB.*

(viii) *Two similarly directed lines with a common perpendicular, is the mid-point of this perpendicular.*

(ix) *Two oppositely directed lines with a common perpendicular, is the line bisecting this perpendicular at right angles.*

(x) *A directed point P and a directed line AB intimate with it, is a horocycle having as an axis the directed line PL perpendicular to AB, the sense of AB relative to L being the same as the sense of the directed point P.*

10. THE GENERALISED ANGLE-BISECTOR AND SIDE-BISECTOR THEOREM.

If α_0, β_0, γ_0 be three directed elements and if λ be the symmetric between β_0 and γ_0, μ the symmetric between γ_0 and α_0, ν the symmetric between α_0 and β_0, then λ, μ, ν are co-intimate.†

Let a_1, b_1, c_1 ; a_2, b_2, c_2 ; a_3, b_3, c_3 be the co-ordinates of α_0, β_0, γ_0, respectively. Then the equations of the symmetrics λ, μ, ν are respectively

$$(a_2-a_3)x+(b_2-b_3)y-(c_2-c_3)z=0$$

$$(a_3-a_1)x+(b_3-b_1)y-(c_3-c_1)z=0$$

$$(a_1-a_2)x+(b_1-b_2)y-(c_1-c_2)z=0$$

Since the determinant

$$\begin{vmatrix} a_2-a_3 & b_2-b_3 & c_2-c_3 \\ a_3-a_1 & b_3-b_1 & c_3-c_1 \\ a_1-a_2 & b_1-b_2 & c_1-c_2 \end{vmatrix}$$

identically vanishes, the theorem is established.

* The locus of points equidistant from two given parallel lines is a line parallel to both. This line is defined to be the middle parallel of the two given lines.

† For a summary of cases see Art. 12, *loc. cit.*

11. THE GENERALISED MEDIAN THEOREM.

If α_+, β_+, γ_+ be directed elements, and λ, μ, ν be the symmetrics between β_+ and γ_-, γ_+ and α_-, α_+ and β_- respectively, then the basic elements $(\alpha\lambda)$, $(\beta\mu)$, $(\gamma\nu)$ are co-intimate.

Let the co-ordinates of α_+, β_+, γ_+ be respectively a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3. Then the equation of λ is

$$(a_2+a_3)\,x+(b_2+b_3)y-(c_2+c_3)z=0$$

and the equation of α is

$$a_1x+b_1y-c_1z=0$$

Hence the equation of $(\alpha\lambda)$ the join of α and λ is

$$(A_2-A_3)x+(B_2-B_3)y-(C_2-C_3)z=0$$

where A_1, B_1 etc., are the minors of the corresponding small letters in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$$

Similarly the equations of $(\beta\mu)$ and $(\gamma\nu)$ are

$$(A_3-A_1)x+(B_3-B_1)y+(C_3-C_1)z=0$$

$$(A_1-A_2)x+(B_1-B_2)y-(C_1-C_2)z=0$$

Since the determinant

$$\begin{vmatrix} A_2-A_3 & B_2-B_3 & C_2-C_3 \\ A_3-A_1 & B_3-B_1 & C_3-C_1 \\ A_1-A_2 & B_1-C_2 & C_1-C_2 \end{vmatrix}$$

vanishes identically, the theorem is established.

12. THE GENERALISED PERPENDICULAR THEOREM.

If α, β, γ be three basic elements then the three elements $\{(\beta\gamma)\alpha\}$, $\{(\gamma\alpha)\beta\}$, $\{(\alpha\beta)\gamma\}$ are co-intimate.

Let

$$a_1x+b_1y-c_1z=0$$

$$a_2x+b_2y-c_2z=0$$

$$a_3x+b_3y-c_3z=0$$

be the equations of α, β, γ.

Then the equation $(\beta\gamma)$ the join of β and γ is

$$(b_2c_3-b_3c_2)x+(c_2a_3-c_3a_2)y-(a_2b_3-a_3b_2)z=0$$

or

$$A_1x+B_1y-C_1z=0$$

where A_1, B_1, etc., are as before.

The equation of $\{(\beta\gamma)\alpha\}$, the join of $(\beta\gamma)$ and α is then

$$(b_1C_1-c_1B_1)x+(c_1A_1-a_1C_1)y-(a_1B_1-b_1A_1)z=0$$

and similar equations may be obtained for $\{(\gamma\alpha)\beta\}$ and $\{(\alpha\beta)\gamma\}$.

Now consider the determinant

$$\begin{vmatrix} b_1C_1-c_1B_1 & c_1A_1-a_1C_1 & a_1B_1-b_1A_1 \\ b_2C_2-c_2B_2 & c_2A_2-a_2C_2 & a_2B_2-b_2A_2 \\ b_3C_3-c_3B_3 & c_3A_3-a_3C_3 & a_3B_3-b_3A_3 \end{vmatrix}$$

The sum of the constituents in the first column is

$$(b_1C_1+b_2C_2+b_3C_3)-(c_1B_1+c_2B_2+c_3B_3)$$

which is zero. Similarly the sum of the elements in every column is zero. Hence the determinant identically vanishes and this establishes our theorem.

A NOTE ON THE STEREOSCOPIC REPRESENTATION OF FOUR-DIMENSIONAL SPACE *

BY

S. MUKHOPADHYAYA

In an address "On the fourth dimension of space" delivered before the Moslem Institute on the 3rd February, 1912, I referred to a stereoscopic device which had suggested itself to me, for visualizing figures in four-dimensional space. It may be mentioned that the possibility of visualizing four-dimensional figures has been predicted by Poincaré.

It is well known that a duplicate picture in plano of a solid figure, taken from two slightly different points of view, when properly looked at through a stereoscope, impresses one with the vividness of a single figure in three dimensions. On the same principle, suitably constructed wire diagrams in three dimensions, whose bases are stereoscopically related, should appear four-dimensional when viewed through a stereoscope. This will only hold for simple Geometrical figures about whose expected appearances in four dimensions the mind has been previously prepared by study and thought of four-dimensional Geometry.

One simple experiment may be easily made by any one. Take a stereoscopic duplicate chart mounted on stiff card-board, containing three white axes at right angles on a black back-ground, some such charts as are enclosed with Airy's Tract on Partial Differential Equations. Stick a couple of equal white pins at the two origins normally to the board. If now the chart be viewed through a stereoscope, four white lines including the pin, will appear to stand out mutually at right angles, which is only possible in four-dimensional space.

* From Bulletin, Cal. Math. Soc., Vol. 4, 1912.

It should be observed that the picture on the retina is a two-dimensional one. The effort at adjusting the optic axes of the eyes in binocular vision gives us the perception of a third dimension. The effort at contracting the crystalline lens for focussing at objects at near distances could give us the perception of a fourth dimension, but this latter adjustment takes place simultaneously with the former and not independently of it from acquired habit of looking at objects in three dimensions and consequently a certain amount of strain on the eyes is experienced when we try to realize through a stereoscope a four-dimensional figure. For the complete realization nevertheless we require more of mental development than organic.

REPLY TO PROF. BRYAN'S CRITICISM *

BY

S. Mukhopadhyay

I

I am glad the subject of a brief note of mine, on the stereoscopic representation of a figure in four dimensions (*Bulletin, C. M. S.*, Vol. IV, 1912-13, page 15) has interested Prof. Bryan. He gives us another method of representation.† From the very imperfect explanations given by him, it is difficult to form a clear conception of his extraordinary pair of stereoscopic pictures. I hope he will impart to us further details.

Apparently his method does not aim at visualizing stereoscopically a four-dimensional figure in all its dimensions, at the same time, as my method does, but only at giving, successively, two three-dimensional aspects of a four-dimensional figure, differing in phase by 90°. If so his method does not go very far.

I thought I had described my method in my note with sufficient clearness. The pair of stereoscopic pictures in my method are not two-dimensional, as is the case with Prof. Bryan's method, but three-dimensional, consisting of a pair of rectilineal figures in space (constructed of wire or thread), standing on an ordinary stereoscopic pair of plane rectilineal figures as bases. The simple experiment I have suggested illustrates the principles of my method. It gives a solution of the problem of four dimensions, by representing before our eyes four lines standing out mutually, at right angles, or at any rate, a close approximation to such a solution.

The principles on which true vision of four dimensions may be possible, stereoscopically or otherwise, have been already set forth by Poincaré. Speaking of complete vision he says (*Science and Hypothesis*, translated by W.J.G., pages 53,54):

* From Bulletin, Cal. Math. Soc., Vol. VI, 1914.

† Bulletin, Cal. Math. Soc., Vol. VI, 1914.

"It has, it is true, exactly three dimensions; which means that the elements of our visual sensations (those at least which concur in forming the concept of extension) will be completely defined if we know three of them; or, in mathematical language, they will be functions of three independent variables. But let us look at the matter a little closer. The third dimension is revealed to us in two different ways: by the effort of accommodation and by the convergence of the eyes. No doubt these two indications are always in harmony; there is between them a constant relation; or, in mathematical language, the two variables which measure these two muscular sensations do not appear to us as independent.........But that is, so to speak, an experimental fact. Nothing prevents us *a priori* from assuming the contrary, and if the contrary takes place, if these two muscular sensations both vary independently, we must take into account one more independent variable, and complete visual space will appear to us as a physical continuum of four dimensions."

In my method I may claim that the independent variation of the two muscular sensations would find ample scope, if we could ever so educate ourselves as to acquire this power of independent variation. My method might be a help towards such an education. At any rate, it would place before our eyes a fairly approximate representation of four-dimensional figures and be useful to us in the study of four-dimensional geometry.

Professor Bryan after all seems to admit that my method is also a possible method of representing stereoscopically a four-dimensional figure, but he says that his method is superior to mine inasmuch as it depends only on the *single* principle of binocular vision, whereas mine requires the *additional* principle of accommodation. *A priori*, it would seem evident that to produce from the two-dimensional picture on the retina a four-dimensional impression, *two* and not *one* independent physiological adaptations of the eyes are indispensably necessary. I do not, however, see any good in further prolonging the controversy between us. Both of us have fairly stated our methods. It would lie with other mathematicians interested in the problems of four dimensions to accept or reject either.

A NOTE ON CURRENT VIEWS OF OPERATIONS THROUGH THE FOURTH DIMENSION *

BY

S. MUKHOPADHYAYA

The object of the present note is first to suggest some rational genesis of a supposed four-dimensionality of our spatial universe and then to examine in the light of this genesis the possibility of certain extraordinary operations which have been currently imagined possible through the fourth dimension.

A universe of space unbounded and Euclidean and of any given number of dimensions can logically exist as a mathematical conception. If we supposed such a space to exist the realm of Nature could not claim the whole of it. The realm of Nature must be closed in the sense that the boundary must belong to it. This is a fundamental hypothesis we will make. It is based on the principle of continuity in Nature. If the realm of Nature were closed by the plane at infinity, the plane at infinity should have a geometry consistent with plane Euclidean geometry, as consistency is the prime attribute of Nature. But the geometry of the plane at infinity is not at all Euclidean.

There would be nothing illogical to suppose that a universal space, unbounded, Euclidean but of *four* dimensions, existed and that the realm of Nature was a self-closed three-dimensional boundary to *some portion* of this four-dimensional universe. This hypothesis gives a wider view of the universe of space and of its relation to Nature-space. A three-dimensional Nature-space by the side of a four-dimensional universe actually existing, dwindles, however, to a filmy nothingness.

A way out of the difficulty is to consider the four-dimensional universe only as a creation of the mind to serve as a scaffolding on which to construct the Non-Euclidean Geometry of Nature-space.

* From Bulletin, Cal. Math. Soc., Vol. IX, 1917.

In fact we may dispense with this scaffolding altogether and make the Non-Euclidean Geometry of Nature-space self-supporting. We return here however again to the three-dimensionality of Nature-space.

Many operations impossible in three-space has been said to be possible through the fourth dimension. For example it has been said that a purse of gold placed in a closed iron safe could be pitch-forked out through the fourth dimension without opening the safe. The possibility of success or otherwise of such an operation would depend on the hypothesis one made between matter and universal space.

Suppose, for example, that the universal space is unbounded, Euclidean and of *five* dimensions. Through every point of our three-dimensional space suppose a circle of variable radius is drawn into this five-dimensional space, no two of the circles being coincident. Suppose further that these circles form a non-intersecting but continuous system and generate a self-closed four-dimensional space necessarily un-Euclidean. The bundle of circles associated with a molecule of matter may be supposed to form an annular system which maintains its identity indissoluble so long as the molecule does so. We may thus have an extended Nature-space of four dimensions, annular, self-closed and un-Euclidean in an Euclidean universal space of five dimensions. This five-dimensional universal space might be looked upon merely as a mental scaffolding to the four-dimensional Nature-space. Each molecule might be supposed extended four-dimensionally along its annulus. Such a four-dimensional Nature-space would be rational but none of the impossible operations in three dimensions would be rendered possible in this amplified field of molecular annuli.

SOME GENERAL THEOREMS IN THE GEOMETRY OF A PLANE CURVE*

BY

S. MUKHOPADHYAYA

Introductory.

The following paper suggests a number of general Theorems in the geometry of elementary plane curves. Indications of proof have been given by the New Methods introduced by the writer. Rigorous proofs have not been attempted. The nomenclatures introduced may appear somewhat novel. They have been found convenient. Besides the paper is meant to appear in a Jubilee Volume, where a certain latitude for novelty may be permissible.

1. Consider a *fixed* continuous plane arc S. Call it the *stem*. The two ends of the stem are A and B. Call A the *lower* end and B the *upper* end. The *positive* direction along the stem is from A to B. The arc is described by a point P moving always in the same sense and not attaining the same position more than once.

At each point P of the stem suppose a tangent exists. The positive direction of this tangent at P is along the positive direction of the stem at P. Suppose this positive direction of the tangent varies in a continuous manner as we proceed up the stem from A to B. The stem is free from cusps and nodes.

If the two points A and B coincide, the stem is *closed* and the point where the two ends meet is the *point of closure*.

An *oval* is a closed stem of which every point may be looked upon as the point of closure. The positive direction along the oval will be taken to be counter-clockwise.

2. Consider a *variable* curve T which crosses the stem S at a limited number of points $P_1, P_2, \ldots, P_s$. Call it the *tendril*.

We will suppose $P_1, P_2, \ldots, P_s$ are arranged in ascending order along the stem, so that P_1 is *above* A, P_2 is above $P_1, \ldots$, and B is above P_s. We may also say A is *below* P_1, P_1 is below $P_2, \ldots, P_s$ is below B.

* From Sir Asutosh Mookerjee Silver Jubilee Volume, No. 2, 1922 (Calcutta University Publication).

We will say that the tendril is *intimate* with the stem at P_1, $P_2, \ldots, P_s$, or that, $P_1, P_2, \ldots, P_s$ is the *range of intimacy* of the tendril with the stem. Two points P_m, P_{m+1} between which no other point of intimacy lies will be called *consecutive* points of intimacy.

In certain cases a selected number of consecutive points of intimacy $P_k, P_{k+1}, \ldots, P_{k+l}$ will be specially called *the points of intimacy* and the remaining points of intimacy which lie above or below these special points of intimacy will be called the points of *sub-intimacy*.

We will suppose the tendril to be a closed branch or a branch extending to infinity on both sides of some well-known algebraic curve of kind K of which the co-efficients are freely or conditionally variable and which does not possess a node or a cusp. The order n of this curve as well as the assign ed conditions to which the co-efficients may be subjected will determine the kind K of the tendril.

The tendril of kind K will have index r if r arbitrary points of intimacy of the tendril with the stem suffice to determine the tendril uniquely.

The tendril may however be defined to pass through a certain number of *fixed* points in the plane besides the r variable points on stem. In general any r arbitrary points lying on the plane in addition to these fixed points, if they exist, will determine the tendril uniquely.

3. The following *conditions of intimacy* of the tendril with the stem will be supposed to hold. When these conditions hold the stem will be called *congenial* to the tendril.

(*i*) The points of intimacy of the tendril with the stem have the same order and sense on the stem as on the tendril.

Suppose $P_1, P_2, \ldots, P_s$ are in ascending or positive order on the stem. Then $P_1, P_2, \ldots, P_s$ will also be in ascending or positive order on the tendril. We will say

The tendril *embraces* the stem in the same order and sense.

(*ii*) The tendril crosses the stem alternately from left to right and right to left.

As we proceed up the stem from A to B we will suppose that there is a continuous region to the right and a continuous region to the left of which the stem is the separating line. The tendril crosses the stem from left to right when it passes from the left region to the right region and it crosses from right to left when it passes from the right region to the left region. Between two consecutive crossings, the tendril, we will suppose, lies wholly in the same region.

A crossing of the stem by the tendril from left to right we will call a *positive* point of intimacy. And a crossing from right to left we will call a *negative* point of intimacy. Hence we may say

The range of intimacy of the tendril with the stem consists of elements of alternately contrary signs.

(*iii*) Two tendrils of kind K and index r cannot have more than $r-1$ points common in the stem or in a certain neighbourhood of the stem.

These $r-1$ points are exclusive of any fixed points through which the tendril may pass by definition. As r is the index of the tendril, two tendrils having r points common will be one and the same.

(*iv*) The tendril varies continuously with the r points of intimacy which suffice to determine it.

The tendril varies continuously in form and position as the r points of intimacy are varied in any continuous manner along the stem. In particular if the r determining points are taken in any interval δ of the stem which tends to vanish, the tendril will tend to a unique limiting form and position. The same may be said if the r determining points are divided into groups which lie in intervals tending simultaneously to vanish. The idea of continuity of variation involves the idea that the tendril does not split up or degenerate or develop nodes or cusps.

(*v*) The number of K points on the stem is limited.

A K point will be defined in the next article.

The stem will contain either no K points or a limited number of K points, separated by finite intervals. If there were an unlimited number of K points on the stem there would exist limiting points of K points on the stem. The existence of these limiting points is impossible as the number of K points is limited.

4. A range of $r+1$ points of intimacy of the tendril of kind K with the stem, taken in order with alternately contrary signs will be called a K *range*. The points of the K range are its *elements*. A K range will be called *positive* or *negative* according as its first element is positive or negative.

If there be other points of intimacy lying between the extreme points of the K range besides those which belong to the K range they will be called *extra* points of the K range. These extra points will necessarily occur in pairs of elements of contrary signs lying between pairs of consecutive elements of the K range, for two consecutive elements of the K range are of contrary signs by definition and consecutive elements of the entire range of intimacy of the

tendril with the stem are also of contrary signs. A K range which does not possess extra points will be called *clear*.

If there be other points of intimacy above or below the extreme points of the K range, they will be called *sub-extra* points.

The $r+1$ elements of the K range together with the extra points when they exist constitute the set of points of intimacy of the K range. The sub-extra points, when they exist, constitute the set of points of sub-intimacy of the K range. The set of points of intimacy of the K range together with the set of points of sub-intimacy constitute the entire range of intimacy of the tendril with the stem.

The interval of the stem, lying between two extreme elements of the K range is called the *interval* of the K range.

A part of the tendril lying between two consecutive points of the range of intimacy will be called a *loop* of intimacy. Loops of intimacy will be alternately on the right and left or left and right sides of the stem. A loop lying on the right will be called positive and a loop lying on the left will be negative.

A neighbourhood of a point O of the stem will be called *upper, lower* or *double* according as the neighbourhood extends to the upper, lower or both sides of O. The unqualified expression *neighbourhood of O* shall always mean a double neighbourhood of O.

A point O of the stem will be called a K point if every neighbourhood of O contains a K range of given sign. The K point would be positive or negative, according as the corresponding K range is positive or negative. A positive K point will be written as $+K$ point and a negative K point will be written as $-K$ point.

A tendril is said to have contact of order p with the stem at O, if in every neighbourhood of O there are $p+1$ consecutive points of intimacy of the tendril with the stem. Thus at a K point, the tendril has contact of order r with the stem.

Imaginary points and so-called coincident points of intimacy do not count in our investigations. Whenever we say that a tendril has contact of order p with the stem at O we imply the actual existence of the set of $p+1$ real and distinct consecutive points of intimacy in every arbitrary neighbourhood of O. The contact position of the tendril is derived as a limit. It does not pre-exist in the logical order of thought. In the contact position, the tendril may be said to have *just left* intimacy with the stem, or we may say that in the contact position the tendril is just on the point of gaining intimacy with the stem. By adopting this point of view we shall avoid saying in

any case that a number of points of intimacy of the tendril with the stem has coincided.

5. One K range is said to be higher than another K range if the elements of the former are higher than the corresponding elements of the latter with possibly some coinciding.

A continuous variation of the elements of a K range will be called a *proper variation* if—

(*i*) during the variation, the elements of the K range remain within the stem ;

(*ii*) the elements of the K range as well as the extra elements of the K range when they exist or are developed maintain their relative order. Any consecutive two may come into as close a neighbourhood as one wishes but do not coincide with or cross each other. Extra elements when they exist or are developed do not disappear ;

(*iii*) sub-extra elements of the K range when they exist or are developed may afterwards disappear, but do not coincide with or cross the extreme elements of the K range.

A proper variation of a K range will be called *elementary* if during the variation $r-1$ elements of the K range remains invariable and the other two elements vary.

An elementary variation will be called an elementary *contraction* if during the variation, the two variable elements continually approach each other.

A K range will be said to undergo a *progressive* contraction if it undergoes a series of elementary contractions in which each element moves in a constant direction or remains stationary during each contraction of the series.

If a set of consecutive elements of a K range are brought together by a proper variation within any arbitrarily small neighbourhood of O, they are said to *congregate* at O. A K point, for example, is a point at which all the $r+1$ elements of a K range congregate.

A set of consecutive elements are said to congregate *beside* O if they are brought into an arbitrarily small upper or lower neighbourhood of O. In the former case we will say they congregate *upside* O and in the latter case *downside* O.

A progressive contraction of a clear K range will be called *simple* if the elements of the K range divide into two groups, a lower and an upper which continually approach each other. The two extreme elements P_1 and P_{r+1} are the first to undergo an elementary contraction till P_1 (or P_{r+1}) congregates beside P_2 (or P_r). The

congregation P_1P_2 and the element P_{r+1} are then made to approach each other by alternate elementary contractions of $P_2.P_{r+1}$ and $P_1.P_{r+1}$ till the congregation P_1P_2 comes beside P_3 or P_{r+1} comes beside P_r. The process is continued in this manner. It will result in congregation of all the elements at a K point unless stopped at some stage. As soon as extra points are developed the process must stop or it may stop when all the elements on one side of an arbitrary fixed point O within the interval has congregated beside O.

One K range is said to *cross* another K range, which is either higher or lower, if the interval of each contain in its interior an extreme element of the other.

Two cross ranges are said to have *external cross contact* if the elements of each range which lie in the common interval of the two cross ranges congregate beside each other, so that the common interval is arbitrarily small.

The cross ranges are said to have *internal cross contact*, if the elements of one range which lie in a non-overlapping part of its interval congregate beside the nearest extreme element of the other range, so that this non-overlapping part is arbitrarily small.

An interval of the stem will be called *free* if it does not contain any K point in its interior.

An interval of the stem will be called *prime* if it contains in its interior only one K point.

An interval of the stem will be called *composite* if it contains in its interior more than one K point.

A K range will be called *prime* if it does not possess any extra elements, neither does it develop any extra elements during any proper variation in its interval. A K range in a prime interval will be prime but the interval of a prime K range is not necessarily prime.

A K range which is not prime will be called *composite*.

Suppose a K range initially clear develops during a simple progressive contraction a pair of extra points. We can now reduce the range by considering the two highest or two lowest points of the range as sub-extra or by considering each of the extreme points of the range as sub-extra. In the first case, the reduction is *unilateral* and in the second case the reduction is *bilateral*. A unilateral reduction is *infra-lateral* or *supra-lateral* according as the two lowest or the two highest elements of the range are reduced.

6. We will now establish some elementary theorems. **The stem will be supposed to be congenial to the tendril.**

Theorem I.—The sign of each element of a K range as well as of each extra element remains invariable during a proper variation.

If any element of the range of intimacy of the tendril with the stem change sign, then every element must change sign at the same time as consecutive elements of the range of intimacy are of contrary sign. This is impossible as the elements of a K range as well as the extra elements of the K range maintain their relative order during a proper variation and do not cross or coincide with each other. If all the elements of a range of intimacy change sign, then all the loops of intimacy change sign and in doing so must coincide with the stem at some stage. But a loop of intimacy cannot coincide with the stem as the number of points common to the tendril and stem is always limited.

The only conceivable way in which an element P of a K range may change sign is when two extra elements are developed indefinitely close to P on either side. This case will be dealt with in the course of demonstration of the next theorem.

Theorem II.—Extra elements of a K range are developed in pairs between consecutive elements of the range.

Consider a K range initially clear of extra elements. The development of an extra element is preceded by the bending down of one of the loops of intimacy on the corresponding interval of the stem giving rise to a contact of the p^{th} order of the tendril with the stem at a point O which is either an interior point or an end point of the interval $P_k\ P_{k+1}$.

First suppose O is an interior point of $P_k\ P_{k+1}$. Then in an arbitrary neighbourhood of O falling within $P_k\ P_{k+1}$ there are developed $p+1$ extra points of intimacy. Now as the signs of P_k, P_{k+1}, originally contrary, continue to be so after the development of the extra points of intimacy by proper variation and as the extra points must obey the law of alternately contrary signs with the elements of the K range, they must be even in number.

Now suppose O is an end-point of $P_k\ P_{k+1}$. Say O is at P_k. Then in an arbitrarily small neighbourhood of P_k there are developed $p+1$ points of intimacy of which one is P_k and the others are extra points. These $p+1$ points lie between P_{k-1} and P_{k+1} which are of the same sign. Consequently $p+1$ must be an odd number. Hence the number of extra points of intimacy developed will be even. This set of $p+1$ points of intimacy will be of alternately contrary sign.

We can identify any of these of a sign contrary to that of P_{s-1} or P_{s+1} as the point P_s, so that between P_s and P_{s-1} as also between P_s and P_{s+1} there will be an even number of extra points of intimacy. If P_s be the lowest element of the K range then we can choose as P_s the lowest possessing suitable sign of the set of $p+1$ points ; so that the new points of intimacy developed will consist of any even number of extra elements and a single or no sub-extra element. The same might be said if the point O were at P_{s+1}.

If the K range be not initially clear then the new extra points will be developed in pairs falling between pairs of consecutive elements of the K range for the old extra points by definition exist in pairs between consecutive points of the K range.

If extra elements are developed simultaneously at each of the $r+1$ points $P_1, P_2, \ldots, P_{r+1}$ of the K range and if the topmost and bottommost points developed have the same signs as P_{r+1} and P_1 respectively then we can identify them with P_{r+1} and P_1 and with suitable identifications at all the other points of the K range, the K range will maintain the signs of its elements inviolate and consequently the number of extra points developed between any two consecutive points of the K range will be even. If however the topmost or bottommost extra point differ in sign from P_{r+1} or P_1 then we can maintain the sign of P_{r+1} or P_1 inviolate by considering this extra point as sub-extra.

Theorem III.—In an elementary variation of a K range the two variable elements of the K range move in opposite directions and in general any two variable elements in the whole range of intimacy of the tendril which have between them no other variable element always move in opposite directions.

First, consider two variable consecutive elements P_s and P_{s+1} of the range of intimacy of the tendril with the stem. If possible suppose in an elementary variation P_s and P_{s+1} receive small displacements in the same direction, say upwards, to P'_s and P'_{s+1} where P'_s lies between P_s and P_{s+1}. Then the loops $P_s\ P_{s+1}$ and $P'_s\ P'_{s+1}$ are of the same sign and the intervals $P_s\ P_{s+1}$ and $P'_s\ P'_{s+1}$ cross each other. Consequently the loops $P_s\ P_{s+1}$ and $P'_s\ P'_{s+1}$ must cross each other at some point. Thus two different tendrils of kind K having $r-1$ points common have another point common which is impossible.

Next, consider two variable elements P_s, P_t of the range of intimacy of the tendril with the stem which have between them only

elements which are invariable. Suppose P_s and P_t are displaced to P'_s and P'_t by an elementary variation. The loops P_sP_{s+1} and $P_{t-1}P_t$ where P_{s+1} and P_{t-1} are invarible elements must lie both within or both without the loops P'_sP_{s+1} and $P_{t-1}P'_t$ for every pair of corresponding loops of two tendrils having r—1 points common on the stem must possess this property. Hence if P'_s lie between P_s and P_{s+1}, then P'_t will lie between P_{t-1} and P_t and if P'_s be below P_s then P_t will lie above P_t. Thus P_s and P_t will be displaced always in the same direction.

Lastly, suppose P_s and P_t are two variable elements of the K range which have between them no other element of the K range or invariable elements of the K range. If no extra elements of the K range lie between P_s and P_t, then the proof already given holds. If any extra elements exist between P_s and P_t, then they will exist in pairs. Suppose there is only one such pair $P_g\,P_{g+1}$. Then if P_s move downwards P_g will move upwards and consequently P_t will move upwards. Similarly if P_s move upwards P_t will move downwards. If there are more than one pair of extra points between P_s and P_t similar proof will hold.

Theorem IV.—In any proper variation of a prime K range it cannot happen that the elements of the K range are all displaced in the same direction or some are displaced in the same direction and the rest are invariable.

Suppose P_1, P_2,.... P_{r+1} are the initial positions of the elements of the K range. Suppose if possible all of them are displaced upwards by a proper variation to new positions P'_1, P'_2,.... P'_{r+1}. Some however may be considered invariable. By a series of elementary variations of the range P'_1, P'_2,.... P'_{r+1} bring down P'_1 down to P_1 while all the other elements move upwards. Again apply a similar method to bring P'_2 down to P_2 while P'_1 remains invariable and all the other elements move upwards. By repetitions of the method all the elements except $P'_rP'_{r+1}$ will have been brought back to their original positions and P'_r and P'_{r+1} will have both moved further upwards from P_r and P_{r+1} which is impossible by Theorem III.

Theorem V.—A prime K range converges to a unique K point.

By a simple progressive contraction the interval of a K range can evidently be made to acquire a sequence of diminishing values converging to zero, each interval lying within the preceding one.

The sequence of intervals define a certain point O on the stem which is common to all the intervals. In every neighbourhood of this point O there is a K range. Therefore the point O is a K point of the same sign as the given K range for a K range maintains its sign during a proper variation.

This K point O is unique. If possible suppose by some other method the K range converges to some other point O' on the stem where O' is above O. Take two sufficiently small neighbourhoods about O and O' which do not overlap. Then there is a K range in each of these neighbourhoods such that one is a proper variation of the other. This is impossible by Theorem IV. as in that case all the elements of the K range about O will have moved upwards to the neighbourhood of O' by a proper variation.

Theorem VI.—A K point cannot at the same time be both positive and negative.

In a positive K range the tendril crosses from left to right at the lowest point of the range. Hence in the limiting form to which the tendril tends as the elements of the K range converge to the corresponding K point, the tendril approaches the stem from the left side. Similarly at a negative K point, the tendril approaches the stem from the right side. Now as the limiting form to which the tendril tends, as the determining points of intimacy approach each other is unique, we see that the given K point cannot at the same time be positive as well as negative.

But it may be argued that at a particular point O, the tendril may have a contact with the stem of order $r+1$. In this case the tendril should have in every arbitrary neighbourhood of O, $r+2$ points of intimacy with the stem. Of these $r+2$ points of intimacy if we take the first $r+1$ we shall have a K range of a given sign, say positive. If we take the last $r+1$ points we shall have a K range which is negative. Consequently it may be argued that at the point O, there exists both a positive and a negative K point. But a little consideration will show that such a contingency is impossible. From a purely geometrical point of view a contact of the $r+1^{th}$ order at O implies the existence of $r+2$ real points of intimacy in every arbitrary neighbourhood of O. Now if we try by a simple progressive contraction to make the first $r+1$ points to converge at O, the $r+2^{th}$ point will be continually moving away from O, so that if the interval in which the $r+2$ points existed at any

moment was arbitrarily contracted it would soon cease to hold the $r+2^{th}$ point.

Again suppose we have an unlimited number of K points in the stem. These will be alternately positive and negative as we shall prove later on. Suppose O is a limiting point of these K points. Then in every neighbourhood of O, there will be a positive K point as well as a negative K point and consequently a positive K range as well as a negative K range. In this case the point O might be called a positive as well as a negative K point. This contingency does not however arise as we have supposed the number of K points on a stem to be always limited. [*Vide* condition (V) of congeniality.]

This theorem is fundamental to our investigations.

Theorem VII.—If a composite K range undergo a progressive contraction with unilateral reductions it will ultimately converge to a K point of the same sign as the original K range.

Suppose we start with a K range initially clear of extra points and apply to it a simple progressive contraction with unilateral reductions whenever a pair of extra points are developed. This unilateral reductions will not alter the sign of the K range. Repeat this process continually. Then a certain stage will be reached after which simple progressive contraction will no further develop extra points.

For if the development of extra points continued indefinitely while the interval of the K range converged to a point O, then in every neighbourhood of O there would be a K range with extra points. This K range with extra points by unilateral and bilateral reductions would give rise to two K ranges with different signs. Consequently the point O would be both a positive and a negative K point which is impossible.

Thus every K point converges by simple progressive contractions with unilateral reductions to at least one K point of the same sign which is interior to its interval. The unilateral reductions we will suppose always supra or always infra although the argument does not require it.

Theorem VIII.—Every K point has a neighbourhood in which the corresponding K range is prime.

Take any prime neighbourhood of K; there must exist a K range of the same sign as that of K in this neighbourhood. This K range will be prime, for if by any proper variation in the prime interval,

a pair of extra points are developed, then by bilateral reduction we shall get a K range of the opposite sign which will converge to a corresponding K point. This latter K point being of a sign different from that of the given K point must be a point different from it. Consequently there are two K points in the same prime neighbourhood which is impossible.

Theorem IX.—The K points of a stem are alternately positive and negative.

Suppose O and O' are two consecutive K points on a stem S, O' being above O. Suppose O is a $+K$ point. Take any prime neighbourhood of O, this neighbourhood will not contain O' as an interior point. Any K range $P_1, P_2, \ldots, P_{r+1}$ in this neighbourhood of O' will be positive. That is, the point P_1 will be positive. Some of the elements of this range will be above O, others below O. We can transfer the element nearest to O on the downside by a simple progressive contraction of the K range in which the remaining elements on the downside of O remains invariable. By repeating this process we can transfer all the elements on the downside of O except the last element to the upside of O.

Take any prime neighbourhood of O' with corresponding K range $P'_1, P'_2, \ldots P'_{r+1}$. We can transfer all the elements $P'_1, P'_2, \ldots, P'_r$ to the downside of O' while P'_{r+1} remains on the upside of O'. Now the interval OO' is free. We can therefore transfer $P_2, P_3, \ldots, P_{r+1}$ to $P'_1, P'_2, \ldots, P'_r$ respectively without development of any further points of intimacy, for in a prime interval there cannot exist more than r points of intimacy. Consequently $P_2, P_3, \ldots, P_{r+1}$ will carry their signs with them when they are transferred to $P'_1, P'_2, \ldots P'_r$. But the tendril is determined by the r points of intimacy. Therefore the signs of P_1 and P'_1 are contrary. And hence the K points O and O' are of contrary signs.

Cor.—In an oval there are always an even number of K points for they are of alternately contrary signs.

Theorem X.—If of two prime K ranges of opposite signs one be above the other, then the point of convergence of the first is above the point of convergence of the second.

The two K ranges being prime and of opposite signs will converge to two distinct and unique K points of opposite signs. If the two K ranges be separate, that is, if every element of the first be above

every element of the second, with possibly the lowest element of the first coinciding with the highest element of the second, then evidently the K point to which the first converges is above the K point to which the second converges, as the K point corresponding to each K range is an interior point of its interval.

It is only in the case where the two K ranges cross each other that the theorem requires proof.

Suppose the first range is P_1, $P_2, \ldots, P_{r+1}$ which is above the second range Q_1, $Q_2, \ldots, Q_{r+1}$. Apply a simple progressive contraction to the range P_1, $P_2, \ldots, P_{r+1}$ till the elements of the range below Q_{r+1} congregate on the downside of Q_{r+1} or the elements above Q_{r+1} congregate on the upside of Q_{r+1}. It may be observed that during this simple progressive contraction of the first range, the first range continues to be above the second range.

In the first case the two ranges will have external cross contact and a progressive contraction applied to the second range will separate the two ranges and the theorem will follow.

In the second case the two ranges will have internal cross contact. Now apply a simple progressive contraction to the second range, till the elements of the second range above P_1 congregate on the upside of P_1 or the elements of the second range below P_1 congregate on the downside of P_1.

In the first case the two ranges will have external cross contact and can be separated by a further simple progressive contraction given to the first range.

In the second case the two ranges will have internal cross contact. By continual application of simple progressive contractions alternately on the two ranges they will either separate or continually contract and converge to a common point O, which will be thus both a positive and a negative K point, which is impossible.

Cor.—If P_1, $P_2, \ldots P_{r+p}$ be $r+p$ consecutive points of intimacy of the tendril with the stem and if the ranges $P_1 \ldots, P_{r+1}$, $P_2 \ldots P_{r+2}$, ... $P_p \ldots P_{r+p}$ be all prime, they will converge to P unique K points of alternately contrary signs.

Theorem XI.—A composite K range converges to a highest and a lowest K point which have the same sign as the original K range.

Suppose we start with a K range initially clear and apply to it progressive simple contraction. At some stage it will develop a pair of extra points. By infra and supra reductions we shall respectively

get two K ranges of the same sign which cross each other, the first being above the second. If we go on applying progressive simple contractions with infra reductions to the first range we shall get the highest K point of the range and if we go on applying progressive simple contractions with supra reduction to the second range we shall get the lowest K point of the range.

If we adopt the method of cross contact explained in Theorem X to the two cross ranges with infra reductions to the first and supra reductions to the second they would always continue to be cross, that is, the first will continue to be higher than the second with a common interval between them or they will separate.

If they do not separate at all then they will ultimately converge to a common K point in every neighbourhood of which there will be two K ranges of the same sign which cross each other of which one is necessarily higher than the other. This is impossible (Theorem IV) as ultimately the neighbourhood will be prime.

Cor. 1.—Every composite K range converges to at least three K points, as between the two extreme K points of the same sign there is a K point of the opposite sign. Theorem IX.

Cor. 2.—If two composite K ranges of contrary signs cross each other, they will converge to at least four K points.

We will now enunciate a general theorem of importance.

Theorem XII.—If an oval tendril of kind K and index r, have $2p$ $(\leq r+1)$ points of intimacy with an oval stem congenial to the tendril, then there will exist on the oval at least $2p$ distinct K points on the stem.

Suppose P_1, $P_2, \ldots, P_{2p}$ are the $2p$ points of intimacy. They form $2p$ successive K ranges $P_1P_2 \ldots P_{r+1}$, $P_2P_3 \ldots P_{r+2}, \ldots$, $P_{2p}P_1 \ldots P_r$ of which any two consecutive ones are of opposite sign and cross each other.

If all the ranges be prime, then by Theorem X. they converge to $2p$ unique K points of alternately contrary signs and the stem will contain exactly $2p$ distinct K points.

If some or all the ranges be composite, the number of K points to which they will converge will be greater.

A GENERAL THEORY OF OSCULATING CONICS—I *

BY

S. MUKHOPADHYAYA.

Formulae and Theorems relating to Osculating Conics are to be found scattered in Text-Books and Journals, but they do not seem to have been treated anywhere in a collective form connected by a general theory.

The methods of deduction of the equations from first principles adopted in this paper may appeal to many as new. Many of the results obtained in this paper will, it is hoped, be found to be also new.

Exclusive use has been made of the method of differentials, as distinguished from that of differential co-efficients, in deducing the fundamental equations. Each of the co-ordinates x and y of any point of the curve have been supposed to be funstions of an independent variable, not expressed. The differential co-efficients, of x and y with respect to the independent variable which we may call t, of any required order, are supposed existing and finite, such that the limits $\Delta t=0$ of $\Delta^n x/(\Delta t)^n$ and $\Delta^n y/(\Delta t)^n$, where $\Delta^n x$ and $\Delta^n y$ are to be interpreted in the sense they are used in the Calculus of Finite Differences, are respectively equal to the n-th differential coefficients of x and y with respect to t, for the necessary values of n.

1. The general equation of a conic, passing through two given points (x, y) and (x_1, y_1), must be of the form

$$\lambda(X-x)(X-x_1)+\mu(Y-y)(Y-y_1)+\nu(X-x)(Y-y_1)$$
$$+\rho(X-x)(Y-y_1)=0 \qquad \ldots \quad (1)$$

as is evident from the number of arbitrary constants involved.

Therefore, the equilateral hyperbola through (x, y) and (x_1, y_1) is of the form

$$\lambda\{(X-x)(X-x_1)-(Y-y)(Y-y_1)\}+\nu(X-x)(Y-y_1)$$
$$+\rho(X-x_1)(Y-y)=0. \qquad \ldots \quad (2)$$

* From Journal of the Asiatic Society of Bengal, New Series, Vol. IV, 1908.

Therefore, the equilateral hyperbola through (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3), is

$$\begin{vmatrix} (X-x)(X-x_1)-(Y-y)(Y-y_1) & (X-x)(Y-y_1) & (X-x_1)(Y-y) \\ (x_2-x)(x_2-x_1)-(y_2-y)(y_2-y_1) & (x_2-x)(y_2-y_1) & (x_2-x_1)(y_2-y) \\ (x_3-x)(x_3-x_1)-(y_3-y)(y_3-y_1) & (x_3-x)(y_3-y_1) & (x_3-x_1)(y_3-y) \end{vmatrix}=0 \quad (3)$$

or,

$$\begin{vmatrix} (X-x)(X-x_1)-(Y-y)(Y-y_1) & (X-x)(Y-y_1) & (Y-y)(x_1-x)-(X-x)(y_1-y) \\ (x_2-x)(x_2-x_1)-(y_2-y)(y_2-y_1) & (x_2-x)(y_2-y_1) & (y_2-y)(x_1-x)-(x_2-x)(y_1-y) \\ (x_3-x)(x_3-x_1)-(y_3-y)(y_3-y_1) & (x_3-x)(y_3-y_1) & (y_3-y)(x_1-x)-(x_3-x)(y_1-y) \end{vmatrix}=0. \quad \ldots \quad (4)$$

Now if (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3) be four consecutive points on a curve, separated by equal infinitesimal increments of the value of the independent variable, then evidently

$$x_1=x+dx,\ x_2=x_1+dx_1,\ x_3=x_2+dx_2.$$

$$\left.\begin{array}{l}\text{Therefore, } x_2=x+dx+d(x+dx)=x+2dx+d^2x,\ x_3=x+2dx \\ +d^2x+d(x+2dx+d^2x)=x+3dx+3d^2x+d^3x\end{array}\right\} \quad (5)$$

with corresponding expressions for y_1, y_2, y_3.

On making substitutions (5) in equation (4), we have, after simplifying the determinant by subtracting three times the second row from the third and ultimately neglecting all infinitesimals of a higher order,

$$\begin{vmatrix} (X-x)^2-(Y-y)^2 & (X-x)(Y-y) & (Y-y)dx-(X-x)dy \\ 2dx^2-2dy^2 & 2dxdy & d^2ydx-d^2xdy \\ 6(d^2xdx-d^2ydy) & 3(d^2ydx+d^2xdy) & d^3ydx-d^3xdy \end{vmatrix}=0. \quad \ldots \quad (6)$$

Equation (6) is the equation of the osculating equilateral hyperbola, at any point (x, y) of a curve.

If the independent variable be x, then $d^2x=0$, $d^3x=0$, and if we write p, q, r for

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},$$

the equation (6) becomes

$$\{(X-x)^2-(Y-y)^2\}(2pr-3q^2)-2(X-x)(Y-y)\{(1-p^2)r+3pq^2\}$$
$$+6\{(Y-y)-(X-x)p\}q(1+p^2)=0. \qquad \dots \quad (7)$$

2. As another illustration of the method of last article, we may determine, in general differentials, the equation of the circle of curvature.

The equation of a circle passing through (x, y), (x_1, y_1), is evidently of the form

$$(X-x)(X-x_1)+(Y-y)(Y-y_1)$$
$$=\lambda\{(Y-y)(x_1-x)-(X-x)(y_1-y)\} \qquad \dots \quad (8)$$

Therefore the equations of a circle passing through any three points, (x, y), (x_1, y_1), (x_2, y_2) is

$$(X-x)(X-x_1)+(Y-y)(Y-y_1)$$
$$=\frac{(x_2-x)(x_2-x_1)+(y_2-y)(y_2-y_1)}{(y_2-y)(x_1-x)-(x_2-x)(y_1-y)}\{(Y-y)(x_1-x)-(X-x)(y_1-y)\} \quad (9)$$

If now (x, y), (x_1, y_1), (x_2, y_2) be three consecutive points on any curve, separated by equal infinitesimal increments of the value of the independent variable, then as in equations (5), $x_1=x+dx$, $x_2=x+2dx+d^2x$, with corresponding expressions for y_1 and y_2.

Therefore, equation (9) gives

$$(X-x)^2+(Y-y)^2=\frac{2(dx^2+dy^2)}{dxd^2y-dyd^2x}\{(Y-y)dx-(X-x)dy\}. \quad \dots \quad (10)$$

Equation (10) is the equation of the circle of curvature. Hence, the co-ordinates of the centre of curvature and the radius of curvature are given by

$$\left.\begin{aligned} X&=x-\frac{(dx^2+dy^2)\,dy}{dxd^2y-dyd^2x} \\ Y&=y+\frac{(dx^2+dy^2)\,dx}{dxd^2y-dyd^2x} \\ \rho&=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y-dyd^2x}. \end{aligned}\right\} \qquad \dots \quad (11)$$

If x be the independent variable equations (11) become

$$\left.\begin{aligned} X &= x - \frac{(1+p^2)p}{q} \\ Y &= y + \frac{(1+p^2)}{q} \\ \rho &= \frac{(1+p^2)^{\frac{3}{2}}}{q}. \end{aligned}\right\} \qquad \dots \quad (12)$$

3. The co-ordinates of the centre of the osculating equilateral hyperbola (7), as determined by differentiating (7) with respect to X and Y, are

$$\left.\begin{aligned} X &= x + \frac{3qr\,(1+p^2)}{(pr-3q^2)^2+r^2} \\ Y &= y + \frac{3q\,(pr-3q^2)(1+p^2)}{(pr-3q^2)^2+r^2}. \end{aligned}\right\} \qquad \dots \quad (13)$$

If R be the radius vector of the osculating equilateral hyperbola, drawn from the centre to the point of osculation, then, from (13),

$$R = \sqrt{(X-x)^2+(Y-y)^2} = \frac{3\,q\,(1+p^2)}{\sqrt{(pr-3q^2)^2+r^2}}. \qquad \dots \quad (14)$$

If P be the perpendicular from centre on the tangent at the point of osculation, then, from (13),

$$P = \frac{p\,(X-x)-(Y-y)}{\sqrt{1+p^2}} = \frac{9q^3\,\sqrt{1+p^2}}{(pr-3q^2)^2+r^2}. \qquad \dots \quad (15)$$

The axis of the equilateral hyperbola bisects the acute angle between R and P. If a be the length of the semi-axis, then

$$a^2 = R.P = \frac{27q^4\,(1+p^2)^{\frac{3}{2}}}{\{(pr-3q^2)^2+r^2\}^{\frac{3}{2}}}. \qquad \dots \quad (16)$$

4. *Theorem I.*—The locus of centres of equilateral hyperbolas osculating a given parabola, is an equal parabola, which is the reflexion of the former on the directrix.

For, taking the parabola to be $y=\frac{x^2}{4a}$, we have $p=\frac{x}{2a}$, $q=\frac{1}{2a}$, $r=0$.

Therefore from (13), $X=x$, $Y=y-2a$ whence the theorem.

Theorem II.—The locus of centres of equilateral hyperbolas, osculating a given central conic, is the inverse of the conic with respect to the director circle. (Noticed by Wolstenholme.)

For, taking the conic to be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, it is easily shewn, by (13), that

$$X=\frac{x\,(a^2+b^2)}{x^2+y^2},\quad Y=\frac{y\,(a^2+b^2)}{x^2+y^2} \qquad \ldots \quad (17)$$

whence the theorem.

5. If an equilateral hyperbola and a parabola both osculate a given curve at a given point they osculate each other, for each of them passes through the same four consecutive points on the curve.

Hence, from Theorem I, we conclude that—(*i*) The directrix of the osculating parabola at a point P of a curve bisects at right angles, the line joining P with the centre Q of the osculating equilateral hyperbola; (*ii*) If O be the middle point of PQ and S the focus of the osculating parabola, then S is the reflexion of O on the tangent at P.

Hence from (13), we easily deduce the equation for the directrix of the osculating parabola to be

$$r(X-x)+(pr-3q^2)\,(Y-y)-\tfrac{3}{2}q\,(1+p^2)=0. \qquad \ldots \quad (18)$$

And if (α, β) be the co-ordinates of the focus S of the osculating parabola, then, from (13), we easily deduce

$$\left.\begin{aligned} \alpha&=x+\frac{3q}{2}\cdot\frac{(1+p^2)\,r-6pq^2}{(pr-3q^2)^2+r^2} \\ \beta&=y+\frac{3q}{2}\cdot\frac{p(1+p^2)r+3q^2(1-p^2)}{(pr-3q^2)^2+r^2}. \end{aligned}\right\} \qquad \ldots \quad (19)$$

The equation, of the osculating parabola itself, is therefore $(X-\alpha)^2+(Y-\beta)^2$

$$=\frac{\{r(X-x)+(pr-3q^2)(Y-y)-\tfrac{3}{2}q(1+p^2)\}^2}{(pr-3q^2)^2+r^2} \qquad \ldots \quad (20)$$

which, after substitutions (19) for α, β, becomes

$$\{(X-x)(pr-3q^2)-(Y-y)r\}^2=18q^3\,\{(Y-y)-p\,(X-x)\}. \qquad \ldots \quad (21),$$

The semi latus rectum (l), of the above parabola, is the perpendicular from the focus (α, β) on the directrix (18). Therefore,

$$l=\frac{27q^5}{\{(pr-3q^2)^2+r^2\}^{\frac{3}{2}}} \qquad \dots \quad (22)$$

It may be noticed here that the focal distance of P and the focal perpendicular on the tangent at P, are respectively $\frac{1}{2}R$ and $\frac{1}{2}P$, given by (14) and (15).

6. If two central conics, one of them being an equilateral hyperbola, osculate a given curve at a given point, then they evidently osculate each other; hence, from Theorem II of article (4), we draw the following conclusions:—

(*i*) The locus of centres of osculating conics, to a given curve at a given point, is a straight line.

For, the given point P and the centre Q, of the osculating equilateral hyperbola are, from equations (17), in one straight line with the centre C, of any other osculating conic. The equation of this line of centres PQ is evidently from (13),

$$(pr-3q^2)\,(X-x)-r\,(Y-y)=0. \qquad \dots \quad (23)$$

(*ii*) The director circles of the osculating conics to a given point of a curve form a co-axial system, having two real limiting points P and Q.

For, $CP.CQ=a^2+b^2$, from equations (17), C being the centre of the osculating conic and therefore of its director circle.

The foregoing conclusions might have been arrived at from simple geometrical considerations. The system of osculating conics, at a given point, have been looked upon, analytically, as having four consecutive points common with the curve. This is not, however, the best way of looking from the geometrical standpoint. Geometrically we may consider the system of osculating conics as having four consecutive tangents common with the curve. Hence—

(*a*) All osculating conics at a given point P of a curve may be conceived as having been inscribed to the same vanishing quadrilateral, formed by four consecutive tangents. Therefore, from well-known properties of a system of conics inscribed to the same quadrilateral, we have,

(b) The locus of centres of conics osculating a given curve at a given point, is a straight line.

(c) The director circles of this system of conics form a co-axial system.

(d) The radical axis of this co-axial system is the directrix of the osculating parabola.

(e) The limiting points of this co-axial system are the given point P and the centre Q of the osculating equilateral hyperbola.

For, the director circle vanishes only if the conic vanishes or is an equilateral hyperbola.

(f) If C be the centre of any osculating conic, then $CP.CQ$ is equal to the square of the radius of the director circle.

(g) If CD be the semi-diameter, conjugate to CP, of the osculating conic whose centre is C, then

$$CP^2+CD^2=a^2+b^2=CP.CQ=CP^2+CP.PQ.$$

Therefore $CD^2=CP.PQ.$ ... (24)

Evidently the locus of D is a parabola whose focus bisects PS, where S is the focus of the osculating parabola.

7. If we compare the values of ρ, R, P, a and l already obtained (12, 14, 15, 16, 22), we notice a number of obvious relations, of which the most remarkable is

$$a^2=l\rho. \qquad \dots \quad (25)$$

Again if ψ be the angle between the normal and line of centres at P,

$$\cos\psi=\frac{P}{R}=\frac{R}{\rho}=\left(\frac{a}{R}\right)^2=\left(\frac{a}{\rho}\right)^{\frac{2}{3}}=\left(\frac{l}{a}\right)^{\frac{2}{3}}=\left(\frac{l}{\rho}\right)^{\frac{1}{3}} \qquad \dots \quad (26)$$

Therefore if $\psi=0$, then $P=R=a=\rho=l$.

N.B.—The angle ψ has been discussed by Transon (Liouville, Vol. VI). It is easily shewn $\tan\psi=p-\dfrac{(1+p^2)r}{3q^2}=\frac{1}{3}\dfrac{d\rho}{ds}$.

8. To determine the axes of any conic of the system we may proceed as follows :—

From the form of the equation of the line of centres (23), the co-ordinates (X, Y) of the centre C, of any osculating conic of the system, can evidently be written as

$$X = x - \frac{3qr}{\lambda},\ Y = y - \frac{3q(pr - 3q^2)}{\lambda} \qquad \ldots \quad (27)$$

where λ is an arbitrary constant.

$$\text{Whence, } CP = 3q\ \{r^2 + (pr - 3q^2)^2\}^{\frac{1}{2}} \cdot \frac{1}{\lambda} \qquad \ldots \quad (28)$$

and by (14) $PQ = \dfrac{3q(1+p^2)}{\{(pr-3q^2)^2 + r^2\}^{\frac{1}{2}}}$.

$$\text{Therefore by (24)} \quad CD^2 = CP.PQ = 9q^2(1+p^2) \cdot \frac{1}{\lambda}. \qquad \ldots \quad (29)$$

The equation of CD is evidently, by (27),

$$(Y - y) - p(X - x) = \frac{9q^3}{\lambda}. \qquad \ldots \quad (30)$$

Therefore if PM be the perpendicular from P on CD,

$$PM = \frac{9q^3}{\lambda(1+p^2)^{\frac{1}{2}}}. \qquad \ldots \quad (31)$$

Hence, if a and b be the semi-axes of the osculating conic,

$$\left.\begin{aligned} a^2 + b^2 &= CP^2 + CD^2 = \frac{9q^2}{\lambda^2}\{r^2 + (pr - 3q^2)^2 + \lambda(1+p^2)\} \\ a^2b^2 &= CD^2 \,.\, PM^2 = 729 \,.\, \frac{q^8}{\lambda^3}. \end{aligned}\right\} \qquad \ldots \quad (32)$$

The equation of the director circle follows from (27) and (32). It is

$$\left\{X - x + \frac{3qr}{\lambda}\right\}^2 + \left\{Y - y + \frac{3q(pr - 3q^2)}{\lambda}\right\}^2$$

$$= \frac{9q^2}{\lambda^2}\{r^2 + (pr - 3q^2)^2 + \lambda(1 + p^2)\}$$

or

$$\lambda\{(X-x)^2+(Y-y)^2\}+9q\ \{(X-x)r+(Y-y)(pr-3q^2)$$

$$-\tfrac{2}{3}q\ (1+p^2)\}=0. \quad \ldots \quad (33)$$

9. To determine the equation of any conic of the system, let V be any point (XY) on the conic and ξ, η its co-ordinates referred to CP and CD, which are conjugate semi-diameters. Draw VH and VK perpendicular from V on CD and CP, respectively.

Then $$\frac{\xi^2}{CP^2}+\frac{\eta^2}{CD^2}=1$$

But $$\frac{\xi^2}{CP^2}=\frac{VH^2}{PM^2}=\frac{\left\{(Y-y)-p\ (X-x)-\dfrac{9q^3}{\lambda}\right\}^2}{\dfrac{(1+p^2).\ 81q^6}{\lambda^2(1+p^2)}} \text{ by (30, 31)}$$

$$=\frac{[\{(Y-y)-p\ (X-x)\}\lambda-9q^3]^2}{81q^6}$$

and $$\frac{\eta^2}{CD^2}=\frac{\eta^2}{VK^2}\cdot\frac{VK^2}{CD^2}=\frac{CP^2}{PM^2}\cdot\frac{VK^2}{CD^2}$$

$$=\frac{9q^2\{r^2+(pr-3q^2)^2\}}{\dfrac{\lambda^2.\ 81q^6}{\lambda^2(1+p^2)}}\cdot\frac{\{(Y-y)r-(X-x)(pr-3q^2)\}^2}{\{r^2+(pr-3q^2)^2\}9q^2(1+p^2).\ \dfrac{1}{\lambda}}$$

by (28, 31, 23, 29)

$$=\frac{\lambda\{(Y-y)r-(X-x)(pr-3q^2)\}^2}{81q^6}.$$

Therefore

$$[\lambda\{(Y-y)-p(X-x)\}-9q^3]^2+\lambda[(Y-y)r-(X-x)(pr-3q^2)]^2$$

$$=81q^6 \quad \ldots \quad (34)$$

or

$$\lambda\{(Y-y)-p(X-x)\}^2+\{(Y-y)r-(X-x)(pr-3q^2)\}^2$$

$$=18q^3\{(Y-y)-p(X-x)\} \quad \ldots \quad (35)$$

which is the general equation of any conic of the system.

$\lambda=0$, it is a parabola.

If $\lambda(1+p^2)+r^2+(pr-3q^2)^2=0$, it is an equilateral hyperbola.

10. The conic of closest contact has evidently for its centre the point common between two consecutive lines of centres. Let X, Y be the co-ordinates of its centre, so that

$$X=x-\frac{3qr}{\lambda},\quad Y=y-\frac{3q(pr-3q^2)}{\lambda}$$

where λ has to be determined.

Then we must have $\frac{dX}{dx}=0$ and $\frac{dY}{dx}=0$, as the two centres corresponding to x, y, λ and $x+dx$, $y+dy$, $\lambda+d\lambda$ must be identical.

Hence $$\frac{dX}{dx}=1-\frac{3(r^2+qs)}{\lambda}+\frac{3qr}{\lambda^2}\quad\frac{d\lambda}{dx}=0$$

$$\frac{dY}{dx}=p-\frac{3(pr^2+pqs-8q^2r)}{\lambda}+\frac{3qr(pr-3q^2)}{\lambda^2}\,.\,\frac{d\lambda}{dx}=0.$$

Eliminating $\frac{d\lambda}{dx}$ between the above two equations, we have

$$\lambda=3qs-5r^2. \qquad \ldots \quad (36)$$

Therefore the co-ordinates of the centre of the conic of closest contact are

$$X=x-\frac{3qr}{3qs-5r^2}\qquad Y=y-\frac{3q(pr-3q^2)}{3qs-5r^2} \qquad \ldots \quad (37)$$

and the equation of the conic of closest contact is

$$(3qs-5r^2)\,\{(Y-y)-p(X-x)\}^2+\{(Y-y)r-(X-x)(pr-3q^2)\}^2$$
$$=18q^3\,\{(Y-y)-p\,(X-x)\}. \qquad \ldots \quad (38)$$

Therefore the conic of closest contact is an ellipse, hyperbola or parabola, according as $3qs-5r^2$ is positive, negative or zero.

11. It may be interesting to deduce the equation of the conic of closest contact directly by the method of differentials.

The general equation of a conic through (x_1, y_1) and (x, y) is of the form, already given (1), *viz.*,

$$\lambda(X-x)(X-x_1)+\mu\,(Y-y)(Y-y_1)+\nu\,(X-x)(Y-y_1)$$
$$+\rho(Y-y)(X-x)=0_1.$$

Therefore the conic through any five points (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), is

$$\begin{vmatrix} (X-x)(X-x_1) & (Y-y)(Y-y_1) & (X-x)(Y-y_1) & (Y-y)(X-x_1) \\ (x_2-x)(x_2-x_1) & (y_2-y)(y_2-y_1) & (x_2-x)(y_2-y_1) & (y_2-y)(x_2-x_1) \\ (x_3-x)(x_3-x_1) & (y_3-y)(y_3-y_1) & (x_3-x)(y_3-y_1) & (y_3-y)(x_3-x_1) \\ (x_4-x)(x_4-x_1) & (y_4-y)(y_4-y_1) & (x_4-x)(y_4-y_1) & (y_4-y)(x_4-x_1) \end{vmatrix} = 0$$

or

$$\begin{vmatrix} (X-x)(X-x_1) & (Y-y)(Y-y_1) & (X-x)(Y-y_1) & (Y-y)(x_1-x)-(X-x)(y_1-y) \\ (x_2-x)(x_2-x_1) & (y_2-y)(y_2-y_1) & (x_2-x)(y_2-y_1) & (y_2-y)(x_1-x)-(x_2-x)(y_1-y) \\ (x_3-x)(x_3-x_1) & (y_3-y)(y_3-y_1) & (x_3-x)(y_3-y_1) & (y_3-y)(x_1-x)-(x_3-x)(y_1-y) \\ (x_4-x)(x_4-x_1) & (y_4-y)(y_4-y_1) & (x_4-x)(y_4-y_1) & (y_4-y)(x_1-x)-(x_4-x)(y_1-y) \end{vmatrix} = 0. \quad \ldots (39)$$

Now if (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) be five consecutive points on a curve, separated by equal infinitesimal increments of the value of the independent variable, then, as in (3),

$$\left.\begin{aligned} & x_1 = x + dx \qquad x_2 = x + 2dx + d^2x \qquad x_3 = x + 3dx + 3d^2x + d^3x \\ & \qquad\qquad x_4 = x + 4dx + 6d^2x + 4d^3x + d^4x \end{aligned}\right\} \ldots (40)$$

with corresponding expressions for y_1, y_2, y_3, y_4.

On making substitutions (40) in (39), we have, after simplification of the determinant by adding to the third row, the second row multiplied by (-3), and to the fourth row, the third row multiplied by (-4) and the second row multiplied by 6, and by ultimately neglecting all higher orders of infinitesimals,

$$\begin{vmatrix} (X-x)^2 & (Y-y)^2 & (X-x)(Y-y) & (Y-y)dx-(X-x)dy \\ 2(dx)^2 & 2(dy)^2 & 2dxdy & d^2ydx-d^2xdy \\ 6dxd^2x & 6dyd^2y & 3dxd^2y+3dyd^2x & d^3ydx-d^3xdy \\ 6(d^2x)^2+8dxd^3x & 6(d^2y)^2+8dyd^3y & 6d^2xd^2y+4(dxd^3y+dyd^3x), & d^4ydx-d^4xdy \end{vmatrix} = 0 \quad \ldots (41)$$

which is the equation of the conic of closest contact in general differentials.

Equation (41) reduces to (38) when the independent variable is x.

12. It is not difficult to extend the method to the direct determination of the equation of the osculating parabola.

The equation of a curve passing through (x, y), (x_1, y_1) which reduces to a parabola if (x, y) and (x_1, y_1) coincide, is evidently of the form

$$\lambda\sqrt{(X-x)(X-x_1)}+\mu\sqrt{(Y-y)(Y-y_1)}$$

$$=\nu\sqrt{(X-x)(y_1-y)-(Y-y)(x_1-x)}.$$

Therefore, the equation of such a curve passing through any four points (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\begin{vmatrix} \sqrt{(X-x)(X-x_1)} & \sqrt{(Y-y)(Y-y_1)} & \sqrt{(Y-y)(x_1-x)-(X-x)(y_1-y)} \\ \sqrt{(x_2-x)(x_2-x_1)} & \sqrt{(y_2-y)(y_2-y_1)} & \sqrt{(y_2-y)(x_1-x)-(x_2-x)(y_1-y)} \\ \sqrt{(x_3-x)(x_3-x_1)} & \sqrt{(y_3-y)(y_3-y_1)} & \sqrt{(y_3-y)(x_1-x)-(x_3-x)(y_1-y)} \end{vmatrix}$$

$$=0. \quad \ldots \quad (42)$$

Now if (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3) be four consecutive points on a curve, then, from (5),

$$\left.\begin{aligned} &\sqrt{(x_2-x)(x_2-x_1)}=\sqrt{(2dx+d^2x)(dx+d^2x)}=\sqrt{2}(dx+\tfrac{3}{4}d^2x) \\ &\sqrt{(x_3-x)(x_3-x_1)}=\sqrt{(3dx+3d^2x+d^2x)(2dx+3d^2x+d^3x)} \\ &\qquad\qquad =\sqrt{6}\,(dx+\tfrac{3}{4}d^2x), \\ &\qquad\qquad \text{etc.} \\ &\sqrt{(y_2-y)(x_1-x)-(x_2-x)(y_1-y)}=\sqrt{d^2y\,dx-d^2x\,dy} \\ &\sqrt{(y_3-y)(x_1-x)-(x_3-x)(y_1-y)} \\ &=\sqrt{3(d^2y\,dx-d^2x\,dy)+(d^3y\,dx-d^3x\,dy)}=\sqrt{3}\sqrt{dx\,d^2y-dy\,d^2x} \\ &\qquad\qquad \times\left(1+\tfrac{1}{6}\frac{d^3y\,dx-d^3x\,dy}{d^2y\,dx-d^2x\,dy}\right) \end{aligned}\right\} \ldots (43)$$

substituting (43) in (42) and simplifying, we have

$$\begin{vmatrix} X-x & Y-y & 3\sqrt{2}(dxd^2y-dyd^2x)^{\frac{1}{2}}\sqrt{(Y-y)\,dx-(X-x)dy} \\ dx & dy & 3(dxd^2y-dyd^2x) \\ d^2x & d^2y & (dxd^3y-dyd^3x) \end{vmatrix} = 0 \quad \ldots \quad (44)$$

$$\text{or, } (Y-y)\,\{dx(d^3ydx-d^3xdy)-3d^2x(d^2ydx-d^2xdy)\}$$

$$-(X-x)\,\{dy(d^3ydx-d^3xdy)-3d^2y(d^2ydx-d^2xdy)\}$$

$$=3\sqrt{2}(d^2ydx-d^2xdy)^{\frac{3}{2}}\sqrt{(Y-y)\,dx-(X-x)dy} \quad \ldots \quad (45)$$

which is the equation of the osculating parabola. It reduces to (21) if x be the independent variable.

From (45) it is evident that the equation of the line of centres is

$$(Y-y)\,\{\,dx(d^3ydx-d^3xdy)-3d^2x(d^2ydx-d^2xdy)\,\}$$

$$=(X-x)\,\{\,dy(d^3ydx-d^3xdy)-3d^2y(d^2ydx-d^2xdy)\,\}. \quad (46)$$

13. The differential equation of a conic is the condition that the conic of closest contact is stationary. We may determine this condition easily.

The condition that any six points (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) may lie on a conic is evidently,

$$\begin{vmatrix} (x_2-x)(x_2-x_1) & (y_2-y)(y_2-y_1) & (y_2-y_1)(x_2-x) & (y_2-y)(x_1-x)-(y_1-y)(x_2-x) \\ (x_3-x)(x_3-x_1) & (y_3-y)(y_3-y_1) & (y_3-y_1)(x_3-x) & (y_3-y)(x_1-x)-(y_1-y)(x_3-x) \\ (x_4-x)(x_4-x_1) & (y_4-y)(y_4-y_1) & (y_4-y_1)(x_4-x) & (y_4-y)(x_1-x)-(y_1-y)(x_4-x) \\ (x_5-x)(x_5-x_1) & (y_5-y)(y_5-y_1) & (y_5-y_1)(x_5-x) & (y_5-y)(x_1-x)-(y_1-y)(x_5-x) \end{vmatrix} = 0 \quad \ldots \quad (47)$$

Now if (x, y), (x_1, y_1), (x_2, y_2), (x_4, y_4), (x_5, y_5) be six consecutive points on a curve, separated by equal infinitesimal increments of the value of the independent variable, then, as in (5),

$$\left.\begin{array}{l} x_1=x+dx \quad x_2=x+2dx+d^2x \quad x_3=x+3dx+3d^2x+d^3x \\ x_4=x+4dx+6d^2x+4d^3x+d^4x \\ x_5=x+5dx+10d^2x+10d^3x+5d^4x+d^5x \\ \text{with corresponding expressions for } y_1, y_2, y_3, y_4, y_5. \end{array}\right\} \quad \ldots \quad (48)$$

On substituting (48) in (47), we have, after simplification of the determinant by adding to the second row, the first row multiplied by -3, to the third row, the second row multiplied by -4, and first row multiplied by 6, and to the fourth row, the third row multiplied by -5, the second row multiplied by 10 and the first row multiplied by -10, and ultimately neglecting all infinitesimals of higher orders,

$$\begin{vmatrix} dx^2 & dy^2 & 2dxdy & dxd^2y-dyd^2x \\ 3dxd^2x & 3dyd^2y & 3(dxd^2y+dyd^2x) & dxd^3y-dyd^3x \\ 3(d^2x)^2+4dxd^3x & 3(d^2y)^2+4dyd^3y & 6d^2xd^2y+4(dxd^3y+dyd^3x), & dxd^4y-dyd^4x \\ 10d^2xd^3x+5dxd^4x & 10d^2yd^3y+5dyd^4y & 10(d^2xd^3y+d^3xd^2y)+5(dxd^4y+dyd^4x). & dxd^5y-dyd^5x \end{vmatrix}=0 \quad \ldots \quad (49)$$

which is therefore the condition that the conic of closest contact at any point of a curve may be stationary.

If the independent variable be x, then equation (49) reduces to

$$40r^3-45qrs+9q^2t=0 \quad \ldots \quad (50)$$

which is the differential equation of the general conic, as has been deduced by Monge.

Methods of simplification of equations (41) and (49) will be given in the next paper.

A GENERAL THEORY OF OSCULATING CONICS—II *

BY

S. Mukhopadhyaya.

Introduction.

Abel Transon in a classical memoir, published in *Liouville's Journal* (Vol. VI, 1841, Researches on the curvature of lines and surfaces), gave the first impulse to the study of *osculating conics* and *higher affections of curvature*.

To him we owe the important discovery, that if O be the middle point of an infinitesimal chord PQ, and T the summit of the arc PQ, then the line OT, in its limiting position, makes an angle δ with the normal, such that $\tan \delta = \frac{1}{3}\frac{d\rho}{ds}$. He calls the line OT, in its ultimate position, *the axis of deviation*, and takes $\tan \delta$ as the measure of *the rate of deviation of the curve from circular form*, or, of *the second affection of curvature*.

The more exact interpretation of $\tan \delta$ † seems, to the present writer, to be what he has called *the partial rate of variation of curvature*, and the formula $\tan \delta = \frac{1}{3}\frac{d\rho}{ds}$ follows at once from this interpretation.

Transon notices that the deviation axis is the locus of centres of osculating conics of four-pointic contact. He determines the centre of the conic of five-pointic contact, as the intersection of two consecutive deviation axes. The distance R of this centre, from the point

* From *Journal A.S.B.*, New Series, Vol. IV, 1908, pp. 497-509.

† *Vide* 'The Geometrical Theory of a Plane Non-cyclic Arc, finite as well as infinitesimal,' *J.A.S.B.*, New Series, Vol. IV, 1908, pp. 391-402.

of contact, he first expresses in terms of p, $\frac{dp}{dw}$, $\frac{d^2p}{dw^2}$, and then reduces to an expression in q, r, s, taking p to be zero. His result is

$$R = \frac{3\rho^2\left\{\left(\frac{d\rho}{dw}\right)^2+9\rho^2\right\}^{\frac{1}{2}}}{4\left(\frac{d\rho}{dw}\right)^2+9\rho^2-3\rho\,\frac{d^2\rho}{dw^2}}$$

$$=\frac{3q\ (r^2+9q^4)^{\frac{1}{2}}}{3qs-5r^2}$$

He gives elegant geometrical constructions for completely determining the osculating parabola and the osculating conic, after tan δ and R have been determined.

His work is quasi-geometrical. His chief aim was to discover 'the second and third affections of curvature.' His discovery of tan δ was beautiful, and he rightly thought he had obtained the *third affection of curvature* when he had determined the value of R, which enabled him to construct the osculating conic.

Professors M. and R. Roberts and J. Wolstenholme have, as isolated problems set in University Papers or published in Collections of Problems, made a number of useful determinations about osculating conics. They have not done, however, any systematic work, and it is not apparent what methods they may have followed in deducing the results. There is strong presumption that they have mainly relied on Transon's researches.

Dr. A. Mukhopadhyaya, in his admirable contributions to the *Journal* of the Asiatic Society of Bengal, more specially in his paper 'On the differential equation of all parabolas,' has treated the subject more methodically, and has deduced and interpreted several important results.

This second paper is based entirely on certain transformations of analytical equations, deduced in determinant forms, in the first paper. The results have been invariably expressed in general differentials. The use made of the quantities P, Q, R, S, etc., will, it is hoped, be found interesting.

14. The general equation of the osculating conic, obtained as equation (41),* namely—

$$\begin{vmatrix} (X-x)^2 & (Y-y)^2 & (X-x)(Y-y) & (Y-y)\,dx-(X-x)dy \\ 2(dx)^2 & 2(dy)^2 & 2dxdy & d^2ydx-d^2xdy \\ 6dxd^2x & 6dyd^2y & 3dxd^2y+3dyd^2x & d^3ydx-d^3xdy \\ 6(d^2x)^2+8dxd^3x & 6(d^2y)^2+8dyd^3y & 6d^2xd^2y+4(dxd^3y+dyd^3x) & d^4ydx-d^4xdy \end{vmatrix}=0$$

is capable of a simple transformation.

If we write

$$\left|\begin{array}{c} (Y-y)dx-(X-x)dy=L \\ (Y-y)d^2x-(X-x)d^2y=M \\ d^2ydx-d^2xdy=Q \\ d^3ydx-d^3xdy=R \\ d^4ydx-d^4xdy=S \\ d^5ydx-d^5xdy=T \\ d^3yd^2x-d^3xd^2y=R' \\ d^4yd^2x-d^4xd^2y=S' \\ dx^2+dy^2=P \\ dxd^2x+dyd^2y=Q_1 \end{array}\right| \qquad (51)$$

then, equation (41) easily transforms into

$$\begin{vmatrix} L^2 & M^2 & LM & L \\ 0 & 2Q^2 & 0 & Q \\ 0 & 0 & -3Q & R \\ 6Q^2 & -8QR' & -4QR & S \end{vmatrix}=0$$

or,

$$\begin{vmatrix} L^2 & M^2-2QL & LM \\ 0 & -2QR & -3Q^2 \\ 6Q^2 & -8QR'-2QS & -4QR \end{vmatrix}=0$$

or,

$$(3QM-RL)^2+(3QS-5R^2+12QR')L^2=18Q^3L.$$

* See J.A.S.B., New Series, Vol. IV, pp. 157-178.

or,

$$\{(Y-y)(3Qd^2x-Rdx)-(X-x)(3Qd^2y-Rdy)\}^2$$
$$+(3QS-5R^2+12QR')\{(Y-y)dx-(X-x)dy\}^2 \qquad (52)$$
$$=18Q^3\{(Y-y)dx-(X-x)dy\}$$

Hence, the osculating conic is an ellipse, hyperbola or parabola, according as

$$3QS-5R^2+12QR'$$

is positive, negative or zero. (53)

15. Again, the condition that a conic may pass through six consecutive points on any curve, obtained as equation (49), namely,

$$\begin{vmatrix} dx^2 & dy^2 & 2dxdy & dxd^2y-dyd^2x \\ 3dxd^2x & 3dyd^2y & 3(dxd^2y+dyd^2x & dxd^3y-dyd^3x \\ 3(d^2x)^2+4dxd^3x & 3(d^2y)^2+4dyd^3y & 6d^2xd^2y+4(dxdy^3+dyd^3x) & dxd^4y-dyd^4x \\ 10d^2xd^3x+5dxd^4x & 10d^2yd^3y+5dyd^4y & 10(d^2xd^3y+d^3xd^2y)+5(dxd^4y+dyd^4x), & dxd^5y-dyd^5x \end{vmatrix}=0$$

likewise transforms easily into

$$\begin{vmatrix} O & Q^2 & O & Q \\ O & O & -3Q^2 & R \\ 3Q^2 & -4QR' & -4QR & S \\ 10QR & -5QS' & 10QR'+5QS & T \end{vmatrix}=0$$

or,

$$\begin{vmatrix} O & R & 3Q \\ 3Q & S+4R' & -4R \\ 10R & T+5S' & 10R'+5S \end{vmatrix}=0$$

or, $$40R^3-45QRS+9Q^2T-90QRR'+45Q^2S'=0 \qquad (54)$$

which is, therefore, the general form of the differential equation of a conic.

16. The conic of four-pointic contact, at any point (x, y) of a given curve, has the first, second and third differentials of x and y, the same as with the given curve, but the fourth and higher differentials arbitrary, and, in general, different from those with the given curve. Hence if we put, in equation (52),

$$3QS-5R^2+12QR'=\lambda \tag{55}$$

where λ is an *arbitrary constant*, we shall have, as the equation of the system of conics, of four-pointic contact, at any point (x, y) of a given curve,

$$\begin{gathered}\{(Y-y)\ (3Qd^2x-Rdx)-(X-x)\ (3Qd^2y-Rdy)\}^2 \\ +\lambda\{(Y-y)\ dx-(X-x)dy\}^2=18Q^3\{(Y-y)dx-(X-x)dy\}^2\end{gathered} \tag{56}$$

Again, if we consider third and higher differentials of x and y arbitrary, and put $\dfrac{R}{3Q}=\mu$, $\dfrac{\lambda}{9Q^2}=\nu$, where μ and ν are *arbitrary constants*, we have as the equation of the system of conics of three-pointic contact, at any point (x, y) of a given curve,

$$\begin{gathered}\{(Y-y)\ (d^3x-\mu dx)-(X-x)\ (d^2y-\mu dy)\}^2 \\ +\nu\{(Y-y)dx-(X-x)dy\}^2=2Q\{(Y-y)dx-(X-x)dy\}\end{gathered} \tag{57}$$

In particular, the equation of the system of parabolas of three-pointic contact is

$$\{(Y-y)(d^2x-\mu dx)-(X-x)(d^2y-\mu dy)\}^2=2Q\{(Y-y)dx-(X-x)dy\} \tag{58}$$

17. It may be interesting to deduce directly the equation of a conic of three-pointic contact from a special form of the equation of a conic passing through three given points.

Let (x, y), (x_1, y_1), (x_2, y_2) be the co-ordinates of any three points P, P_1, P_2, and let

$$\left.\begin{aligned} L &\equiv (Y-y_1)(x_1-x)-(X-x_1)(y_1-y) \\ M &\equiv (Y-y_1)(x_2-x_1)-(X-x_1)(y_2-y_1) \\ N &\equiv (Y-y)(x_2-x)-(X-x)(y_2-y) \end{aligned}\right\} \tag{59}$$

be the equations of the lines PP_1, P_1P_2 and PP_2, respectively. Then

$$\left.\begin{aligned} M-L &\equiv (Y-y_1)(x_2-2x_1+x)-(X-x_1)(y_2-2y_1+y) \\ M+L &\equiv (Y-y_1)(x_2-x)-(X-x_1)(y_2-y) \\ L+M-N &\equiv (y_2-y)(x_1-x)-(x_2-x)(y_1-y) \end{aligned}\right\} \qquad (60)$$

Now, the equation of a conic through P, P_1, P_2 can evidently be written in the form

$$\lambda LM-\mu N(M-L)+(M-L)^2-(M+L)(M+L-N)=0$$

where λ and μ are arbitrary constants, for, it is the same as

$$\lambda\, LM-\mu(MN-NL)-(4LM-MN-NL)=0$$

which circumscribes $\quad L=0,\ M=0,\ N=0$

Thus, the general equation of a conic, through three given points, is of the form

$$\begin{aligned} &\lambda\{(Y-y_1)(x_1-x)-(X-x_1)(y_1-y)\}\{(Y-y_1)(x_2-x_1) \\ &\qquad -(X-x_1)(y_2-y_1)\} \\ &-\mu\{(Y-y)(x_2-x)-(X-x)(y_2-y)\}\{(Y-y_1)(x_2-2x_1+x) \\ &\qquad -(X-x_1)(y_2-2y_1+y)\} \\ &+\{(Y-y_1)(x_2-2x_1+x)-(X-x_1)(y_2-2y_1+y)\}^2 \\ &-\{(Y-y_1)(x_2-x)-(X-x_1)(y_2-y)\}\{(y_2-y)(x_1-x) \\ &\qquad -(x_2-x)(y_1-y)\}=0 \end{aligned} \qquad (61)$$

Now if (x, y), (x_1, x_2), (y_1, y_2) be consecutive points on a curve then

$$\left.\begin{aligned} x_1=x+dx,\ x_2=x_1=dx_1=x+2dx+d^2x \\ y_1=y+dy,\ y_2=y_1+dy_1=y+2dy+d^2y \end{aligned}\right\}$$

Therefore (61) becomes

$$\begin{aligned} &\lambda\{(Y-y)dx-(X-x)dy\}^2-2\mu\{(Y-y)dx-(X-x)dy\}\{(Y-y)d^2x \\ &\qquad\qquad -(X-x)d^2y\} \\ &+\{(Y-y)d^2x-(X-x)d^2y\}^2-2Q\{(Y-y)dx+(X-x)dy\}=0 \end{aligned}$$

Or, $\{(Y-y)(d^2x-\mu dx)-(X-x)(d^2y-\mu dy)\}^2$

$$+\nu\{(Y-y)dx-(X-x)dy\}^2=2Q\{(Y-y)dx-(X-x)dy\}$$

where $\nu=\lambda-\mu^2$. This equation is the same as (57).

18. Again, the general equation of a cubic through three given points (x, y), (x_1, y_1), (x_2, y_2) can evidently be written in the form

$$\begin{aligned}
&\alpha(X-x)(X-x_1)(X-x_2)+\beta(Y-y)(Y-y_1)(Y-y_2)\\
&+\gamma(X-x)(Y-y_1)(Y-y_2)+\delta(Y-y)(X-x_1)(X-x_2)\\
&+\lambda\{(Y-y_1)(x_1-x)-(X-x_1)(y_1-y)\}\{(Y-y_1)(x_2-x_1)\\
&\qquad-(X-x_1)(y_2-y_1)\}\\
&-\mu\{(Y-y)(x_2-x)-(X-x)(y_2-y)\}\\
&\{(Y-y_1)(x_2-2x_1+x)-(X-x_1)(y_2-2y_1+y)\}\\
&+\{(Y-y_1)(x_2-2x_1+x)-(X-x_1)(y_2-2y_1+y)\}^2\\
&-\{(Y-y_1)(x_2-x)-(X-x_1)(y_2-y)\}\\
&\{(y_2-y)(x_1-x)-(x_2-x)(y_1-y)\}=0
\end{aligned}\tag{62}$$

which contains the necessary terms and the necessary number of arbitrary constants.

Therefore, the cubic of three-pointic contact at any point (x, y) of a curve, is of the form

$$\begin{aligned}
&\alpha(X-x)^3+\beta(Y-y)^3+\gamma(X-x)(Y-y)^2+\delta(Y-y)(X-x)^2\\
&+\lambda\{(Y-y)dx-(X-x)dy\}^2\\
&-2\mu\{(Y-y)dx-(X-x)dy\}\{(Y-y)d^2x-(X-x)d^2y\}\\
&+\{(Y-y)d^2x-(X-x)d^2y\}^2-2Q\{(Y-y)dx-(X-x)dy\}=0
\end{aligned}\tag{63}$$

In general, the equation of a curve of the n^{th} degree, which has three-pointic contact with a given curve at the origin will have the portion below third degree, of the form

$$\begin{aligned}
&\lambda\{Ydx-Xdy\}^2-2\mu\{Ydx-Xdy\}\{Yd^2x-Xd^2y\}\\
&+\{Yd^2x-Xd^2y\}^2-2Q\{Ydx-Xdy\}=0
\end{aligned}\tag{64}$$

19. It is easy to deduce from the general equation of a conic of three or four-pointic contact, that of a four or five-pointic contact, and the method is a useful one.

For example, the general equation of a parabola of three-pointic contact is (58)

$$\{(Y-y)(d^2x-\mu dx)-(X-x)(d^2y-\mu dy)\}^2$$
$$=2Q\{(Y-y)dx-(X-x)dy\}.$$

If this parabola meet the curve again at an adjacent point (X, Y), corresponding to the value $t+\tau$ of the independent variable t then

$$\left.\begin{aligned} X&=x+dx+\frac{1}{1.2}d^2x+\frac{1}{1.\,2.\,3}d^3x+\text{etc.} \\ Y&=y=dy+\frac{1}{1.\,2}d^2y+\frac{1}{1.\,2.\,3.}d^3y+\text{etc.} \end{aligned}\right\} \qquad (65)$$

where d^nx and d^ny stand for $\dfrac{d^nx}{dt^n}.\ \tau^n$ and $\dfrac{d^ny}{dt^n}.\ \tau^n$, respectively.

Substituting (65) in (58) and remembering that μ is an infinitesimal of first order, we have

$$(-Q-\tfrac{1}{2}\mu Q)^2=2Q\{\tfrac{1}{2}Q+\tfrac{1}{6}R\}$$

or,
$$\mu=\frac{R}{3Q}.$$

Again, to determine λ, so that we may get the conic of five-pointic contact, from the system of four-pointic (56),

$$\{(Y-y)(3Qd^2x-Rdx)-(X-x)(3Qd^2y-Rdy)\}^2$$
$$+\lambda\{(Y-y)dx-(X-x)dy\}^2=18Q^3\{(Y-y)dx-(X-x)dy\}.$$

Substitute (65) in (56), and remembering that λ is an infinitesimal of order eight, we have

$$(-3Q^2-\tfrac{1}{2}QR+\tfrac{1}{2}QR'-\tfrac{1}{6}R^2)^2+\lambda\{\tfrac{1}{2}Q+\tfrac{1}{6}R\}^2$$
$$=18Q^3\{\tfrac{1}{2}Q+\tfrac{1}{6}R+\tfrac{1}{24}S\}$$

or, $$9Q^4+3RQ^3+\tfrac{5}{4}R^2Q^2-3R'Q^3+\tfrac{1}{4}\lambda Q^2$$

$$=9Q^4+3Q^3R+\tfrac{3}{4}Q^3S$$

or, $$\lambda=3QS-5R^2+12QR'$$

20. Equation (56) can be written as

$$\{(Y-y)\,(3Qd^2x-Rdx)-(X-x)\,(3Qd^2y-Rdy)\}^2$$

$$+\lambda\left\{(Y-y)dx-(X-x)dy-\frac{9Q^3}{\lambda}\right\}^2=\frac{81Q^6}{\lambda}$$

whence,

$$(Y-y)\,(3Qd^2x-Rdx)-(X-x)\,(3Qd^2y-Rdy)=0 \tag{66}$$

and $$(Y-y)dx-(X-x)dy=\frac{9Q^3}{\lambda} \tag{67}$$

are the Equations of two conjugate diameters.

Equation (66) gives the diameter through the point of contact, and as it is independent of λ, it represents the locus of centres of all conics of four-pointic contact at the given point.

Equation (67) gives the diameter parallel to the tangent at (x,y).

The intersection of (66) and (67) is the centre, whose co-ordinates are

$$X=x+\frac{3Q(3Qd^2x-Rdx)}{\lambda} \qquad Y=y+\frac{3Q(3Qd^2y-Rdy)}{\lambda} \tag{68}$$

The osculating semi-diameter CP is given by

$$CP^2=\frac{9Q^2}{\lambda^2}\{(3Qd^2x-Rdx)^2+(3Qd^2y-Rdy)^2\}$$

$$=\frac{9Q^2\{9Q^4+(3QQ_1-RP)^2\}}{\lambda^2 P} \tag{69}$$

For,
$$(3Qd^2x-Rdx)^2+(3Qd^2y-Rdy)^2$$
$$=9Q^2\{(d^2x)^2+(d^2y)^2\}-6QR\{dxd^2x+dyd^2y\}$$
$$+R^2(dx^2+dy^2)$$
$$=9Q^2\frac{Q^2+Q_1{}^2}{\mathrm{P}}-6QRQ_1+R^2\mathrm{P}$$
$$=\frac{9Q^4+(3QQ_1-RP)^2}{\mathrm{P}} \tag{70}$$

If ψ be the angle between the normal and line of centres (66), called the angle of *aberrancy*, then evidently

$$\left.\begin{aligned}\tan\psi&=\frac{3QQ_1-RP}{3Q^2}\\ \cos\psi&=\frac{3Q^2}{\{9Q^4+(3QQ_1-RP)^2\}^{\frac{1}{2}}}\\ \sin\psi&=\frac{3QQ_1-RP}{\{9Q^4+(3QQ_1-RP)^2\}^{\frac{1}{2}}}\end{aligned}\right\} \tag{71}$$

If a and b be the semi-axes of the conic (56), then, evidently,

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{\lambda}{81Q^6}\{(3Qd^2x-Rdx)^2+\lambda dx^2+(3Qd^2y-Rdy)^2+\lambda dy^2\}$$
$$=\frac{\lambda}{81Q^6\mathrm{P}}\{9Q^4+(3QQ_1-RP)^2+\lambda P^2\}$$
$$\frac{1}{a^2b^2}=\frac{\lambda^2}{81^2Q^{12}}\ [\{(3Qd^2x-Rdx)^2+\lambda dx^2\}\{(3Qd^2y-Rdy)^2+\lambda dy^2\}$$
$$-\{(3Qd^2x-Rdx)(3Qd^2y-Rdy)+\lambda dxdy)\}^2]$$
$$=\frac{\lambda^3}{81^2.Q^{12}}\{(3Qd^2y-Rdy)dx-(3Qd^2x-Rdx)dy\}^2$$
$$=\frac{\lambda^3}{27^2Q^8}$$

Therefore, $a^2+b^2=\frac{9Q^2}{\lambda^2P}\{9Q^4+(3QQ_1-RP)^2+\lambda P^2\}$ (72)

$$ab=\frac{27Q^4}{\lambda^{\frac{3}{2}}}$$

If CD be the diameter conjugate to CP, then from (69) and (72)

$$\left.\begin{aligned}
CD^2=a^2+b^2-CP^2&=\frac{9Q^2P}{\lambda}\\
\frac{CP^2}{CD^2}&=\frac{9Q^4+(3QQ_1-RP)^2}{\lambda P^2}\\
\frac{CD^2}{CP}&=\frac{3QP^{\frac{3}{2}}}{\{9Q^4+(3QQ_1-RP)^2\}^{\frac{1}{2}}}=\rho\cos\psi
\end{aligned}\right\} \tag{73}$$

The equation of the director circle, deduced from (68) and (72), is

$$\lambda\{(X-x)^2+(Y-y)^2\}-6Q\{(X-x)\,(3Qd^2x-Rdx)$$

$$+(Y-y)\,(3Qd^2y-Rdy)+\tfrac{3}{2}QP\}=0 \tag{74}$$

Thus the director circles of the system of conics of four-pointic contact, form a co-axial system, of which the radical axis is

$$(X-x)(3Qd^2x-Rdx)+(Y-y)(3Qd^2y-Rdy)+\tfrac{3}{2}QP=0 \tag{75}$$

This radical axis is the directrix of the osculating parabola.

21. The condition that the osculating conic may be an equilateral hyperbola is $a^2+b^2=0$. Therefore, from (72)

$$\left.\begin{aligned}
\lambda&=-\frac{9Q^4+(3QQ_1-RP)^2}{P^2}\\
\text{and } \alpha^2&=\frac{27Q^4P^3}{\{9Q^4+(3QQ_1-RP)^2\}^{\frac{3}{2}}}=\rho^2\cos^3\psi
\end{aligned}\right\} \tag{76}$$

where α is the semi-axis of the osculating equilateral hyperbola.

The co-ordinates of the point, where the normal at the point of contact meets the equilateral hyperbola again, are found to be

$$\left.\begin{aligned} X&=x+\frac{2Pdy}{Q} \\ Y&=y-\frac{2Pdx}{Q} \end{aligned}\right\} \quad (77)$$

But the co-ordinates of the centre of curvature are (11)

$$X=x-\frac{Pdy}{Q} \qquad Y=y+\frac{Pdx}{Q}$$

Therefore, the osculating equilateral hyperbola meets the normal again, towards the convex side of the curve, at a distance from the point of contact equal to twice the radius of curvature.

Again, as the co-ordinates (77) do not involve higher differentials than the second, we conclude that all equilateral hyperbolas of three-pointic contact pass through the same point (77).

Further, as two consecutive osculating equilateral hyperbolas may be conceived to possess three consecutive points common, they intersect again at (77), and, therefore, the *envelope* of the further branch of the osculating equilateral hyperbola is the locus of the point given by (77).

22. The equation of the osculating parabola, obtained from (56) by putting $\lambda=0$ is

$$\{(Y-y)(3Qd^2x-Rdx)-(X-x)(3Qd^2y-Rdy)\}^2$$
$$=18Q^3\{(Y-y)\,dx-(X-x)dy\} \quad (78)$$

The diameter through point of contact is (66)

$(Y-y)(3Qd^2x-Rdx)-(X-x)(3Qd^2y-Rdy)=0$ and the directrix is (75)

$$(Y-y)(3Qd^2y-Rdy)+(X-x)(3Qd^2x-Rdx)+\tfrac{9}{2}QP=0,$$

The co-ordinates of the point of intersection, of the diameter through point of contact with directrix, are

$$\left.\begin{aligned} X_1 &= x - \tfrac{3}{2}QP^2 \frac{3Qd^2x - Rdx}{9Q^4 + (3QQ_1 - RP)^2} \\ Y_1 &= y - \tfrac{3}{2}QP^2 \frac{3Qd^2y - Rdy}{9Q^4 + (3QQ_1 - RP)^2} \end{aligned}\right\} \quad (79)$$

If (α, β) be the focus, then the join of (α, β) and (X_1, Y_1) is bisected at right angles by the tangent at (x, y), hence

$$\left.\begin{aligned} &\alpha = X_1 - udy \quad \beta = Y_1 + udx \\ &\text{where } u = \frac{9Q^3P}{9Q^4 + (3QQ_1 - RP)^2} \end{aligned}\right\} \quad (80)$$

The semi-latus rectum (l) is the perpendicular from focus on the directrix. Therefore

$$l = \frac{27Q^5 P^{\frac{3}{2}}}{\{9Q^4 + (3QQ_1 - RP)^2\}^{\frac{3}{2}}} = \rho \cos^3\psi \quad (81)$$

The focal distance of (x, y) is equal to the distance of (x, y) from directrix

$$= \frac{\tfrac{3}{2}Q\, P^{\frac{3}{2}}}{\{9Q^4 + (3QQ_1 - RP)\}^{\frac{1}{2}}} = \frac{\rho}{2}\cos\psi. \quad (82)$$

The axis passes through (α, β) and is, therefore,

$$(Y-y)(3Qd^2x - Rdx) - (X-x)(3Qd^2y - Rdy)$$

$$= \frac{9Q^3P\,(3QQ_1 - RP)}{9Q^4 + (3QQ_1 - RP)^2} \quad (83)$$

The normal at the point of contact meets the axis (83) at

$$X = x - udy \quad Y = y + udx \quad (84)$$

The distance of this point, from point of contact, is

$$uP^{\frac{1}{2}} = \frac{9Q^3P^{\frac{3}{2}}}{9Q^4 + (3QQ_1 - RP)^2} = \rho\cos^2\psi \quad (85)$$

The co-ordinates of the intersection of the directrix with the normal at the point of contact are

$$X = x + \frac{Pdy}{2Q} \quad Y = y - \frac{Pdx}{2Q} \tag{86}$$

Therefore the directrix of the osculating parabola meets the normal, towards the convex side of the curve, at a distance from the point of contact equal to half the radius of curvature.

Again, as the co-ordinates (86) do not involve higher differentials than the second, we conclude that the directrices of all parabolas of three-pointic contact, pass through the same point (86).

Further, as two consecutive parabolas of four-pointic contact, may be conceived to possess three consecutive points common, their directrices meet at (86), and therefore, the *envelope* of the directrix of the osculating parabola is the locus of the point (86).

23. If a and b be the semi-axis of any ellipse of the system of conics of four-pointic contact (56), then from (72)

$$\frac{a}{b} + \frac{b}{a} = \frac{1}{3\lambda^{\frac{1}{2}} Q^2 P}\{9Q^4 + 3(QQ_1 - RP)^2 + P^2\lambda\}$$

$$= \frac{3Q^2}{P\lambda^{\frac{1}{2}}}\sec^2\psi + \frac{P\lambda^{\frac{1}{2}}}{3Q^2} \tag{87}$$

But $\left(\frac{a}{b} + \frac{b}{a}\right)^2 = 4 + \frac{e^4}{1-e^2}$.

Therefore $\frac{a}{b} + \frac{b}{a}$ is a minimum when e is a minimum.

Hence, the ellipse of minimum eccentricity of the system (56) is determined by

$$\left.\begin{aligned} \lambda &= \frac{9Q^4 + (3QQ_1 - RP)^2}{P^2} \\ \frac{a}{b} + \frac{b}{a} &= \frac{2}{\cos\psi} \end{aligned}\right\} \tag{88}$$

Therefore, the centre of the osculating ellipse, of minimum eccentricity, is a point, on the line of centres, towards the concave side, at the same distance, from the point of contact, as the centre of the osculating equilateral hyperbola. Here, evidently $CP=CD=\rho \cos \psi$.

Again, if λ_1 and λ_2 corresponds to equal values of the eccentricity, and, therefore, to equal values of $\frac{a}{b}$, then from (87)

$$\sqrt{\lambda_1 \lambda_2} = \frac{9Q^4 + (3QQ_1 - RP)^2}{P^2} \tag{89}$$

Therefore, if C, C_1, C_2 be the centres of the ellipse of minimum eccentricity and of any two ellipses of equal eccentricity, then, from (69)

$$C_1P. \quad C_2P = CP^2 \tag{90}$$

where P is the point of contact.

Analogous results hold for the system of hyperbolas of four-pointic contact.

If Q be the centre of the osculating equilateral hyperbola, and Q_1, Q_2 the centres of any two osculating hyperbolas whose asymtotic angles are supplementary, then we can prove in the same way

$$Q_1P. \, Q_2P = QP^2 \tag{91}$$

Again, if (a_1, b_1) and (a_2, b_2) be semi-axes corresponding to λ_1 and λ_2, then by (72)

$$a_1 \, b_1 = \frac{27Q^4}{\lambda_1^{\frac{3}{2}}} \qquad a_2 b_2 = \frac{27Q^4}{\lambda_2^{\frac{3}{2}}}$$

$$\text{Therefore, } a_1 \, b_1 \, a_2 \, b_2 = \frac{27^2 \, Q^8 P^6}{\{9Q^4 + (3QQ_1 - RP)^2\}^3} = a^4 \tag{92}$$

where a is the semi-axis of the osculating equilateral hyperbola.

24. The system of simple binomial differential quantities P, Q, R, S, T, Q_1, R', S', which have been introduced in the preceding investigations, can, of course, be taken with any independent variable. Of the eight quantities only the first five may be

looked upon as primary, and the rest as dependent auxiliaries. If we take x as the independent variable, then dx is constant, and therefore, d^2x, d^3x, d^4x, d^5x all vanish. The quantities P, Q, R, S, T, Q_1 are, in this case, equal to $(1+p^2)dx^2$, qdx^3, rdx^4, sdx^5, tdx^6, $pqdx^3$, respectively. R' and S' evidently vanish.

If we take the arc (s) as the independent variable, then

$$P = dx^2 + dy^2 = ds^2 = \text{constant}$$

Therefore, $Q_1 = dxd^2x + dyd^2y = \frac{1}{2}dP = 0$

$$(d^2x)^2 + (d^2y)^2 = \frac{Q^2+Q_1^2}{P} = \frac{Q^2}{P} \tag{93}$$

Again $dQ_1 = (d^2x)^2 + (d^2y)^2 + dx\ d^3x + dy\ d^3y = \frac{1}{2}\ d^2P = 0$

Therefore,
$$dx\ d^3x + dy\ d^3y = -\frac{Q^2}{P} \tag{94}$$

Also,
$$dx\ R' - d^2x\ R + d^3x\ Q = 0$$
$$dy\ R' - d^2y\ R + d^3y\ Q = 0$$

Therefore $PR' - RQ_1 + (dx\ d^3x + dy\ d^3y)\ Q = 0$

Hence
$$R' = \frac{Q^3}{P^2} \tag{95}$$

Also,
$$S' = dR' = \frac{3Q^2R}{P^2} \tag{96}$$

The general differential equation (54) of the conic, if s be the independent variable, therefore, becomes

$$40\ R^3 + 9Q^2T = 45\ QR\left(S - \frac{Q^3}{P^2}\right) \tag{97}$$

Again let ρ, ρ', ρ'', ρ''' be the radius of curvature and its three successive differentials, on the supposition that the arc is the independent variable.

Then by (11), (95) and (96)

$$Q = P^{\frac{3}{2}}\frac{1}{\rho},\ R = dQ = -\frac{P^{\frac{3}{2}}}{\rho^2},\ R' = \frac{P^{\frac{5}{2}}}{\rho^3}\ ,\ S' = 3\,\frac{P^{\frac{5}{2}}\rho'}{\rho^4} \tag{98}$$

$$\left.\begin{aligned} \text{Also } S+R' &= dR = P\left(\frac{2\rho'^2}{\rho^3}-\frac{\rho''}{\rho^2}\right) \\ T+2S' &= d^2R = P^{\frac{3}{2}}\left(-\frac{6\rho'^3}{\rho^4}+\frac{6\rho'\rho''}{\rho^3}-\frac{\rho'''}{\rho^3}\right) \end{aligned}\right\} \tag{99}$$

By the above substitutions (98), (99) any expression in P, Q, R, S, etc., can be readily converted into another in P, ρ, ρ', ρ'' and ρ'''.

$$\text{Thus, } 9Q^4+(3QQ_1-PR)^2=\frac{P^6}{P^4}\left(9+\frac{\rho'^2}{P}\right) \tag{100}$$

$$3QS-5R^3+12QR'=\frac{P^6}{\rho^4}\left(9+\frac{\rho'^2}{P}-\frac{3\rho\rho''}{P}\right) \tag{101}$$

$$40\,R^3-45\,QRS+9\,Q^3T-90\,QRR'+45\,Q^2S'$$

$$=-\frac{P^{\frac{9}{2}}}{\rho^6}\left\{4\rho'^3-9\,\rho\,\rho'\,\rho''+9\,\rho^2\,\rho'''+36\,P\,\rho'\right\} \tag{102}$$

Therefore the differential equation of a conic in ρ and s is

$$4\rho'^3-9\,\rho\,\rho'\,\rho''+9\,\rho^2\,\rho'''+36\,P\,\rho'=0$$

or,

$$4\left(\frac{d\rho}{ds}\right)^3-9\rho\frac{d\rho}{ds}\,\frac{d^2\rho}{ds^2}+9\rho^2\,\frac{d^3\rho}{ds^3}+36\frac{d\rho}{ds}=0 \tag{103}$$

ON RATES OF VARIATION OF THE OSCULATING CONIC *

BY

S. MUKHOPADHYAYA.

1. If D^r stand for $\left(\frac{d}{dt}\right)^r$ where t is any independent variable, and Q_{rs}, for $D^r x\, D^s y - D^r y\, D^s x$, where x and y are given analytic functions of t, then the equation of the osculating conic at any point (x, y) of the corresponding plane curve may be written as

$$\{(Y-y)A-(X-x)B\}^2+\Gamma\{(Y-y)Dx-(X-x)Dy\}^2$$
$$=18Q_{12}^3\ \{(Y-y)Dx-(X-x)Dy\}$$

where $\quad A=3Q_{12}D^2x-Q_{13}Dx \qquad B=3Q_{12}D^2y-Q_{13}Dy$

$$\Gamma=3Q_{12}Q_{14}-5Q_{13}^2+12Q_{12}Q_{23}$$

and the condition, that the osculating conic passes through six consecutive points, is $\triangle=0$ where

$$\triangle=40Q_{13}^3-45Q_{12}Q_{13}Q_{14}+9Q_{12}^2Q_{15}-90Q_{12}Q_{13}Q_{23}+45Q_{12}^2Q_{24}.$$

These results have been elsewhere deduced from first principles. *Vide* "A General Theory of Osculating Conics," *Journal, Asiatic Society of Bengal*, Vol. IV, Nos. 4 and 10, 1908. The following method, however, is more simple.

If (x, y) be any given point of the plane curve and (X, Y) any other point on it, the corresponding values of the independent variable being t and $t+\tau$, then

$$X-x=Dx\tau+\frac{1}{2!}D^2x.\ \tau^2+\frac{1}{3!}D^3x.\ \tau^3+\text{etc.}$$

$$Y-y=Dy.\ \tau+\frac{1}{2!}D^2y.\ \tau^2+\frac{1}{3!}D^3y.\ \tau^3+\text{etc.}$$

* From Bulletin, Calcutta Mathematical Society, Vol. I, 1909, pp. 125-130.

Therefore, $$(Y-y)Dx-(X-x)Dy$$

$$=\frac{1}{2!}\,Q_{12}\tau^2+\frac{1}{3!}\,Q_{13}\tau^3+\frac{1}{4!}\,Q_{14}\tau^4+\frac{1}{5!}Q_{15}\tau^5+\text{etc.}$$

and $$(Y-y)D^2x-(X-x)D^2y$$

$$=-Q_{12}\tau+\frac{1}{3!}Q_{23}\tau^3+\frac{1}{4!}Q_{24}\tau^4+\frac{1}{5!}Q_{25}\tau^5+\text{etc.}$$

whence it is shewn

$$\{(Y-y)A+(X-x)B\}^2+\Gamma\{Y-y)Dx-X-x)Dy\}^2$$

$$-18Q_{12}^3\,\{(Y-y)Dx-(X-x)Dy\}=-\frac{1}{60}Q_{12}\triangle\tau^5+\text{etc.}$$

Hence $$\{(Y-y)A-(X-x)B\}^2+\Gamma\{(Y-y)Dx-(X-x)Dy\}^2$$

$$-18Q_{12}^3\{(Y-y)Dx-(X-x)Dy\}=0$$

meets the given curve at five consecutive points at (x, y), determined by $\tau^5=0$. If, however, $\triangle=0$, the $\tau^6=0$, and the point (x, y) is a *sextactic* point on the given curve.

2. If ξ, η be the co-ordinates of the centre of the osculating conic at (x, y), then it is easily shewn

$$\xi=x+\frac{3Q_{12}A}{\Gamma}\qquad \eta=y+\frac{3Q_{12}B}{\Gamma}$$

To calculate $D\xi$ and $D\eta$, we have

$$\begin{aligned}DA&=D(3Q_{12}D^2x-Q_{13}Dx)\\&=3Q_{12}D^3x+2Q_{13}D^2x-(Q_{23}+Q_{14})Dx\\&=5Q_{13}D^2x-(4Q_{23}+Q_{14})Dx\end{aligned}$$

since $Dx.\,Q_{23}-D^2x.\,Q_{13}+D^3x.\,Q_{12}=0$

Therefore $$D\left(\frac{A}{Q_{12}^{\frac{5}{3}}}\right)=\frac{5Q_{13}D^2x-(4Q_{23}+Q_{14})Dx}{Q_{13}^{\frac{5}{3}}}-\frac{5A\,Q_{13}}{3Q_{12}^{\frac{8}{3}}}$$

$$=-\frac{\Gamma Dx}{3Q_{12}^{\frac{8}{3}}}$$

Similarly $D\left(\dfrac{B}{Q_{12}^{\frac{5}{3}}}\right)=-\dfrac{\Gamma Dy}{3Q_{12}^{\frac{8}{3}}}$

Again $$D\left(\frac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)=\frac{3Q_{12}D\Gamma-8\Gamma Q_{13}}{Q_{12}^{\frac{11}{3}}}=\frac{\triangle}{3Q_{12}^{\frac{11}{3}}}$$

Therefore,

$$D\xi=Dx+3D\frac{\dfrac{A}{Q_{12}^{\frac{5}{3}}}}{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}}=Dx+3\frac{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}D\dfrac{A}{Q_{12}^{\frac{5}{3}}}-\dfrac{A}{Q_{12}^{\frac{5}{3}}}D\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}}{\left(\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^2}$$

$$=-\frac{A\triangle}{\Gamma^2}$$

Similarly $$D\eta=-\frac{B\triangle}{\Gamma^2}$$

If we call the locus of (ξ, η) the curve of aberrancy, and σ the arcual length of the curve of aberrancy, then

$$D\sigma=\{(D\xi)^2+(D\eta)^2\}^{\frac{1}{2}}=\frac{(A^2+B^2)^{\frac{1}{2}}\triangle}{\Gamma^2}$$

So that if $\triangle=0$, then $D\sigma=0$, a result upon which Dr. A. Mukhopadhyaya has based an elegant interpretation of the differential equation of the general conic. (*Vide Journal, Asiatic Society of Bengal*, Vol. LVIII, Part II, page 185.)

3. If a and b be the semi-axes of the osculating conic, then it can be shewn that

$$ab=\frac{27Q_{12}^4}{\Gamma^{\frac{3}{2}}}=27\left(\frac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^{-\frac{3}{2}}$$

It is hence evident that $\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}$ is an invariant of the point (x, y),

i.e., independent of the particular independent variable t, as also of the origin and direction of the axes of co-ordinates,

Again $D(ab) = -\frac{3}{2}.27.\left(\frac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^{-\frac{5}{2}} D\,\frac{\Gamma}{Q_{12}^{\frac{8}{3}}} = -\frac{27}{2}\,\frac{Q_{12}^{3}\triangle}{\Gamma^{\frac{5}{2}}}$

Therefore $$\frac{D(ab)}{Q_{12}^{\frac{1}{3}}} = -\frac{27}{2}.\frac{\dfrac{\triangle}{Q_{12}^{4}}}{\left(\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^{\frac{5}{2}}}$$

But $$\frac{D(ab)}{Q_{12}^{\frac{1}{3}}} = \frac{D(ab)}{\{(Dx)^2+(Dy)^2\}^{\frac{1}{2}}}.\frac{\{(Dx)^2+(Dy)^2\}^{\frac{1}{2}}}{Q_{12}^{\frac{1}{3}}} = \frac{d(ab)}{ds}.\,\rho^{\frac{1}{3}}$$

where s is arcual length, and ρ the radius of curvature of the given curve at (x, y).

Hence $\dfrac{D(ab)}{Q_{12}^{\frac{1}{3}}}$ is an invariant, Therefore also $\dfrac{\triangle}{Q_{12}^{4}}$ is an invariant.

Again if r and r_1 denote two conjugate semi-diameters of the osculating conic of which r passes through the point of contact, then

$$r^2 = (\xi - x)^2 + (\eta - y)^2 = 9Q_{12}^{2}.\frac{A^2+B^2}{\Gamma^2} = 9\,\frac{\dfrac{A^2+B^2}{Q_{12}^{\frac{10}{3}}}}{\left(\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^2}$$

Hence we see that $\dfrac{A^2+B^2}{Q_{12}^{\frac{10}{3}}}$ is also an invariant of the point (x, y).

It is easily shewn from the equation of the osculating conic that

$$R^2 = r^2 + r_1^2 = 9Q_{12}^{2}\left\{\frac{A^2+B^2}{\Gamma^2} + \frac{(Dx)^2+(Dy)^2}{\Gamma}\right\}$$

where R is the radius of the director circle of the osculating conic.

Therefore $r_1^2 = 9Q_{12}^{2}.\ \dfrac{(Dx)^2+(Dy)^2}{\Gamma}$

To calculate $D(r^2)$ and $D(r_1^2)$.

we have $$D\,\frac{A^2+B^2}{Q^{\frac{10}{3}}} = -\frac{2\Gamma(ADx+BDy)}{3Q_{12}^{\frac{13}{3}}} = -\frac{2\Gamma C}{3Q_{12}^{\frac{13}{3}}}$$

where $ADx+BDy \equiv C$. $\dfrac{C}{Q_{12}^{2}}$ is evidently an invariant.

Also $$D\,\frac{(Dx)^2+(Dy)^2}{Q_{12}^{\frac{2}{3}}} = \frac{2C}{2Q_{12}^{\frac{5}{3}}}$$

Since $$D\,\frac{Dx}{Q_{12}^{\frac{1}{3}}} = \frac{A}{3Q_{12}^{\frac{4}{3}}} \text{ and } D\left(\frac{Dy}{Q_{12}^{\frac{1}{3}}}\right) = \frac{B}{3Q_{12}^{\frac{4}{3}}}$$

We have, therefore,

$$D(r^2) = 9D\left\{\frac{\dfrac{A^2+B^2}{Q_{12}^{\frac{10}{3}}}}{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}}\right\} = -6Q_{12}\left(\frac{C}{\Gamma} + \frac{A^2+B^2}{\Gamma^3}\triangle\right)$$

$$D(r_1^2) = 9D.\frac{\dfrac{(Dx)^2+(Dy)^2}{Q_{12}^{\frac{2}{3}}}}{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}} = 6Q_{12}\left(\frac{C}{\Gamma} - \frac{(Dx)^2+Dy)^2}{2\Gamma^2}\triangle\right)$$

Therefore

$$D(R^2) = D(r^2) + D(r_1^2) = -6Q_{12}\left\{\frac{A^2+B^2}{\Gamma^3} + \frac{(Dx)^2+(Dy)^2}{2\Gamma^2}\right\}\triangle$$

4. If θ be the angle which an axis of the osculating conic makes with the x-axis, then it is easily seen that

$$\tan 2\theta = \frac{2AB + 2\Gamma DxDy}{A^2 - B^2 + \Gamma\{(Dx)^2 - (Dy)^2\}} = \frac{M}{N}$$

where

$$M=\frac{2AB+2\Gamma DxDy}{Q_{12}^{\frac{10}{3}}} \quad N=\frac{A^2-B^2+\Gamma\{(Dx)^2-(Dy)^2}{Q_{12}^{\frac{10}{3}}}$$

To calculate $D(\theta)$, we have

$$D\frac{AB}{Q_{12}^{\frac{10}{3}}}=-\frac{\Gamma(BDx+ADy)}{3Q_{12}^{\frac{13}{3}}} \quad D\frac{DxDy}{Q_{12}^{\frac{2}{3}}}=\frac{BDx+ADy}{3Q_{12}^{\frac{5}{3}}}$$

Therefore $$DM=\frac{2\triangle DxDy}{3\,Q^{\frac{13}{3}}}$$

Again $$D\frac{A^2-B^2}{Q_{12}^{\frac{10}{3}}}=-\frac{2\Gamma(ADx-BDy)}{3Q_{12}^{\frac{13}{3}}}$$

$$D\frac{(Dx)^2-(Dy)^2}{Q^{\frac{2}{3}}}=\frac{2(ADx-BDy)}{3Q_{12}^{\frac{5}{3}}}$$

Therefore $$DN=\frac{\{(Dx)^2-(Dy)^2\}\triangle}{3Q_{12}^{\frac{13}{3}}}$$

so that $$D\tan 2\theta=\frac{NDM-MDN}{N^2} \text{ or } 2D(\theta)=\frac{NDM-MDN}{M^2+N^2}$$

But $$NDM-MDN=\frac{2\triangle[(A^2-B^2)\,DxDy-AB\{(Dx)^2-(Dy)^2\}]}{3Q_{12}^{\frac{23}{3}}}$$

$$=\frac{2\triangle(ADx+BDy)(ADy-BDx)}{3Q_{12}^{\frac{23}{3}}}=-\frac{2\triangle C}{Q_{12}^{\frac{17}{3}}}$$

and $$M^2+N^2=\frac{[A^2+B^2-\Gamma\{(Dx)^2+(Dy)^2\}]^2+4C^2\Gamma}{Q_{12}^{\frac{20}{3}}}$$

Therefore $$D(\theta)=-\frac{\triangle CQ_{12}}{[(A^2+B^2)-\Gamma\{(Dx)^2+(Dy)^2\}]^2+4C^2\Gamma}$$

But $$D\left(\frac{C}{Q_{12}^{2}}\right)=\frac{A^2+B^2-\Gamma\{(Dx)^2+(Dy)^2\}}{3Q_{12}^{3}}$$

Therefore
$$D(\theta)=\frac{-\triangle CQ_{12}}{9Q_{12}\left[D\dfrac{C}{Q_{12}^{2}}\right]+4C^{2}\Gamma}$$

$D\theta$ becomes indeterminate, if $C=0$ and also $D\dfrac{C}{Q_{12}^{2}}=0$, which are easily shewn to be the conditions that the osculating conic is a circle. For it can be shewn that

$$D\left(\frac{a}{b}+\frac{b}{a}\right)=\frac{-\triangle D\dfrac{C}{Q_{12}^{2}}}{6\Gamma^{\frac{3}{2}}}$$

which shews that $\dfrac{a}{b}+\dfrac{b}{a}$ has a minimum value when $D\dfrac{C}{Q_{12}^{2}}=0$.

But since
$$\left(\frac{a}{b}+\frac{b}{a}\right)^{2}=4+\frac{e^{4}}{1-e^{2}}$$

where e is eccentricity of the osculating conic, we conclude that

$$D\frac{C}{Q_{12}^{2}}=0$$

is the condition that the osculating conic has minimum eccentricity.

Again if δ be the angle between the line of centres and normal to the curve, then evidently

$$\tan\delta=\frac{C}{3Q_{12}^{2}}$$

so that $C=0$ is the condition that δ vanishes.

It may be pointed out here, in passing, that the apparent way of interpreting the singularity, when the osculating conic reduces to a circle, by saying that three consecutive circles of curvature coincide, is meaningless, unless we can shew that, in the *immediate neighbourhood* of such a singularity, a circle meets the curve in five distinct points. It may be shewn, from geometrical considerations, that such is not the case. In fact, such an interpretation of the singularity would imply the coincidence of an in-cyclic point with an ex-cyclic one, which is not possible.

We may, however, interpret the singularity by saying that when the conic reduces to a circle, two singularities of *different* kinds coincide. These are

$$D\left(\frac{a}{b}+\frac{b}{a}\right)=0 \text{ and } \delta=0$$

If ϕ be the eccentric angle of the osculating conic, at the point of contact, then it is easily shewn,

$$\tan 2\phi=\frac{2\,ab\tan\delta}{r^2-r_1^{\,2}}=\frac{2\Gamma^{\frac{1}{2}}C}{A^2+B^2-\Gamma\{(Dx)^2+(Dy)^2\}}=\frac{2\Gamma^{\frac{1}{2}}C}{3Q_{12}^{\,3}D\left(\frac{C}{Q_{12}^{\,3}}\right)}$$

Therefore $$D(\theta)=-\frac{\triangle Q_{12}\sin^2 2\phi}{4C\,\Gamma}$$

NOTE ON T. HAYASHI'S PAPER ON THE OSCULATING ELLIPSES OF A PLANE CURVE *

BY

S. MUKHOPADHYAYA.

The properties of osculating ellipses which Professor T. Hayashi discusses in his paper † as well as other interesting properties were given by me, believed for the first time, in a paper published by the Calcutta Mathematical Society, Vol. I, No. I, 1909, and reviewed by Professor P. Montel in conjunction with other papers on Finite Geometry, in the Bulletin des Sciences Mathematiques, 1924, Part I. The results were deduced by me in an extremely simple but rigorous manner by a method which was introduced by me in that paper. Two out of many theorems proved in that paper are quoted below to bear out my contention.

If we define an elementary non-sextactic arc AB to be one which has no sextactic point in it, except it may be at the two extremities A and B, the following theorems have been proved to hold, supposing the arc to be of an elliptic nature, that is, the conic through any five points in it is always an ellipse.

Prop. VI. If O_1, O_2, O_3, O_4, O_5 be any five points on such arc then the area of the ellipse $O_1O_2O_3O_4O_5$ will continuously increase (or decrease) if the points be shifted in any manner along the arc in the same direction, provided the order of the points be maintained and the points be never so far separated from one another that the elliptic arc $O_1O_2O_3O_4O_5$ exceeds the semi-ellipse.

Prop. X. If any five points being taken in order, O_1, O_2, O_3, O_4, O_5 on such arc AB, the ellipse $O_1O_2O_3O_4O_5$ cuts in at O_1 and O_5, then the osculating ellipse at A falls entirely within the osculating ellipse at B.

These theorems, it may be noted, are in some respects more general than those given by Professor Hayashi.

The minimum numbers of cyclic and sextactic points on an elementary oval were also first given by me in this paper.

* From Rendiconti del Circolo Matematico di Palermo, t. LI (1927).

† Rendiconti del Circolo Matematico di Palermo, t. L (1926), pp. 419-422.

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SOME OPINIONS ON S. MUKHOPADHYAYA'S WORK

Professor J. Hadamard, Paris: " My interest in your *new methods in the geometry of a plane arc*, which I had expressed in 1909 in an (anonymous) note in the *revue generale des sciences*, has far from diminished since that time.

Precisely at my seminaire or colloquium of the college de France, we have reviewed such subjects and all my auditors and colleagues have been keenly interested in your way of researches which we all consider as one of the most important roads opened to Mathematical Science."

Professor F. Engel, Geissen: " I am surprized over the beautiful *new* calculations on the right-angled triangles and three-right-angled quadrilaterals (in hyperbolic geometry)....Your analogies in the Gaussian Pentagramma Mirificum are highly remarkable."

Professor W. Blaschke, Hamburg: " I am much obliged to you for your kind sending of your beautiful geometrical work. When, as I hope, a new edition of my Lessons in Differential Geometry comes out, I shall not forget to mention that you were the *first* to give the beautiful theorems on the numbers of Cyclic and Sextactic points on an oval."

Professor F. Cajori, California: " I congratulate you upon your success in research. If ever I have the time and opportunity to revise my **History** of Mathematics I shall have occasion to refer to your interesting work."

Professor T. Hayashi, Japan: " Sincerely I congratulate your success on *New Methods in Geometry*, specially on the *new concept of intimacy*.

Professor A. R. Forsyth, London: " Your papers connected with analytical and differential geometry are valuable and interesting."

Professor L. Godeaux, Liége: " A first reading (of your papers) has seized my grand interest. As I have written, I intend making an exposition of these questions early to my students of *Géométrie supérieure*, an exposition to which I reckon to join that of the works of M. Juel."